# Software Implementation of Vinberg's Algorithm for Integral Hyperbolic Lattices

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More detailed description is published in [5].

## 1 Introduction

In 1972 (see [12]), Vinberg proposed an algorithm constructing the fundamental polyhedron of the group $O_r(L)$ for a given lattice $L$ and, thereby, determining whether or not $L$ is reflective. This algorithm has turned out to be useful in both classifying reflective lattices and describing automorphism groups and moduli varieties of K3-surfaces.

Efforts to implement Vinbergs algorithm on a computer have been made since the 1980s, but all of them dealt with particular lattices, usually determined by diagonal quadratic forms. Mentions of such programs can be found, e.g., in the papers [7] by Bugaenko, [11] by Scharlau and Walhorn, [6] by Nikulin, and [1] by Allcock. But the programs themselves have not been published; the only exception is Nikulins paper, which contains a program code for lattices of several different special forms. The only known implementation published together with a detailed documentation is Guglielmettis 2016 program processing diagonal quadratic forms with square-free invariant factors over several ground fields. Guglielmetti used this program in his thesis to classify diagonal quadratic forms with small diagonal elements (see [9]). This program works fairly efficient in all dimensions in which reflective lattices exist.

In this paper, we present an original implementation of Vinbergs algorithm for arbitrary integral (with the ground field $\mathbb{Q}$) hyperbolic lattices subject to no constraints. The project is written in the Sage computer algebra system (see [10]) and is available on the Internet (see [4]). At present, the program works effectively for $2 \leq n \leq 5$. Thus, it turns out to be useful, e.g., for solving the open problem of classifying reflective lattices in the dimension $n = 3$; it has already been successfully applied by the first-named author to obtain partial classification results. We plan to optimize the program so as to make it efficient for $n \leq 10$.

## 2 Choice of a Basic Point

The program is fed by an integer square matrix $G$ of order $n + 1$ determining an inner product of signature $(n, 1)$ in the hyperbolic lattice $L = [G]$ (here $[C]$ denotes the quadratic lattice with inner product determined in some basis by the symmetric matrix $C$). To determine the basic point $v_0$, the program seeks a change of coordinates over $\mathbb{Q}$ such that, in the new coordinates, the matrix of inner product is diagonal and its first diagonal element is negative. The primitive lattice vector proportional to the first vector of the new basis is taken for $v_0$.

## 3 Construction of the Fundamental Cone

Suppose that $v_0$ belongs to an $m$-face of the fundamental cone of the group $O_r(L)_{v_0}$, $1 \leq m \leq n$. Since the outward normals $a_1, \ldots, a_m$ to the faces of the fundamental cone are perpendicular to $v_0$ in $\mathbb{E}^{n,1}$, it follows that they lie in the Euclidean space $V_1 = \langle v_0 \rangle^{\perp}$. However, the lengths of these normals are bounded, and their number is finite; they can be determined by using the coroutine described in Sec. 6 of this paper.

The fundamental cone is constructed as follows. First, we take the cone $C$ equal to the entire space $V$ , and then we perform the following procedure for each root in $V_1$ in turn: if the mirror $H_a$ of a given root $a$ intersects the interior of $C$, then we replace the cone $C$ by its intersection with the half-space $H_a^- = \{v \mid (a, v) \leq 0\}$. This is a finite procedure, and it outputs the fundamental domain of the group $O_r(L)_{v_0}$ as $C$.

## 4 Decomposition of the Lattice Roots

**Proposition 1** *The lattice $L = (L \cap V_1) \oplus \mathbb{Z}v_0$ is a finite-index sublattice of $L$, and $L/L$ is a cyclic group whose order $s = |L/L'|$ divides the number $(v_0, v_0)$.*

**Proof.** The image of each element of $L$ under factorization by $L'$ is uniquely determined by its inner product with $v_0$ modulo $(v_0, v_0)$. ∎

**Corollary 1** *There exist vectors $w_1, \ldots, w_s \in L$ for which $L = \sqcup(w_i + L')$. In particular, each vector $a \in L$ has a unique representation in the form*

$$a = a_0 v_0 + v_1 + w_t,$$

*where $a_0 \in \mathbb{Z}$, $v_1 \in L \cap V_1$, and $1 \leq t \leq s$.*

Let us write the distance minimality condition (3) for Vinbergs algorithm in the above decomposition. Setting $\ell = (v_0, v_0) \in \mathbb{Z}_{<0}$ and $k = (a, a)$, we obtain

$$\sinh \rho(H_a, v_0) = \frac{|(a, v_0)|}{\sqrt{(a, a)(v_0, v_0)}} = \frac{|a_0 \ell + (w_t, v_0)|}{\sqrt{k\ell}}.$$

Therefore, the distance from the mirror $H_a$ to the basic point is determined by the integer $a_0$, the component $w_t$, and the length of the root $a$ itself. But there are only finitely many possibilities for the length of the root and the number of the component $w_t$. Thus, all roots are grouped into sets determined by triples $(a_0, w_t, k)$, which we totally order according to the distance from the mirror $H_a$ to the point $v_0$.

## 5 Determination of Roots

After the fundamental polyhedral cone is found, the program searches through the roots according to the total order on the triples $(a_0, w_t, k)$ introduced above. For each triple, the vector a is determined by the component $v_1 \in V_1$ and the condition $(a, a) = k$; the component is found by the coroutine described in Sec.6. Thus, for each triple, a root satisfying conditions (1) and (2) is found in finite time. The found root satisfies condition (3) by virtue of the ordering method of the triples. The program adds each found root $a_j$ to the system of roots found previously and checks whether the system of roots $(a_1, \ldots, a_j)$ forms the system of outward normals for a Coxeter polyhedron of finite volume. There exist several programs for verifying the finiteness of volume, including Bugaenkos program mentioned above and Guglielmettis[1] program (see [8]), which we have used in this work.

Thus, the program successively determines roots which are outward normals of the sought-for polyhedron $P$. If the roots found at some step determine a polyhedron of finite volume, then the program terminates.

## 6 The Coroutine for Solving Quadratic Diophantine Equations

The computational complexity of Vinbergs algorithm is associated almost entirely with solving a quadratic Diophantine equation of the form

$$xAx^t + Bx^t + c = 0,$$

where $A$ is a positive definite integer symmetric matrix of order $n$, $B$ is an integer row of length $n$, $x = (x_1, \ldots, x_n)$ is a row of unknowns, and $c \in \mathbb{Z}$. In our project, this problem was solved by using a Python coroutine acting by induction on $n$. Namely, since the solutions lie on an ellipsoid, it follows that $x_n$ can take only finitely many integer values. The coroutine finds them by the method of conjugate gradients; for each value of $x_n$, the problem reduces to an $(n-1)$-dimensional problem in the hyperplane $\{x_n = \text{const}\}$. For $n = 1$, the coroutine finds the integer roots of a quadratic trinomial.

## 7 Testing the Program for known Examples of Reflective Lattices

The program was tested for many lattices of rank 3, 4, and 5. For rank 3, examples were taken from Nikulins list of reflective hyperbolic lattices (see [6]), and for ranks 4 and 5, from Scharlau and Walhorns list of reflective maximal isotropic hyperbolic lattices (see [11]); lattices of rank 4 were also taken from classification papers of Vinberg (see [15]) and of the first-named author of this paper (see [2] and [3]). In all of the tests performed by us, the results yielded by the program coincided precisely with those of calculations once

[1] See https://github.com/rguglie1/CoxIter.

| $L$ | # faces | $t$ (sec) |
|---|---|---|
| $[-1] \oplus A_3$ | 4 | 0,7 |
| $[-2] \oplus A_3$ | 5 | 1,9 |
| $[-3] \oplus A_3$ | 5 | 1,0 |
| $[-4] \oplus A_3$ | 4 | 0,66 |
| $[-5] \oplus A_3$ | 6 | 1,56 |
| $[-6] \oplus A_3$ | 6 | 1,5 |
| $[-8] \oplus A_3$ | 7 | 1,72 |
| $[-9] \oplus A_3$ | 9 | 79,5 |
| $[-10] \oplus A_3$ | 12 | 1,72 |
| $[-12] \oplus A_3$ | 5 | 1.02 |
| $[-15] \oplus A_3$ | 12 | 28,7 |
| $U \oplus [36] \oplus [6]$ | 15 | 56,6 |

| $L$ | # faces | $t$ (sec) |
|---|---|---|
| $[-1] \oplus [1] \oplus A_2$ | 4 | 0,6 |
| $[-2] \oplus [1] \oplus A_2$ | 6 | 0,8 |
| $[-3] \oplus [1] \oplus A_2$ | 5 | 0,6 |
| $[-4] \oplus [1] \oplus A_2$ | 5 | 1,02 |
| $[-5] \oplus [1] \oplus A_2$ | 7 | 1,9 |
| $[-6] \oplus [1] \oplus A_2$ | 8 | 1,2 |
| $[-7] \oplus [1] \oplus A_2$ | 11 | 19,2 |
| $[-8] \oplus [1] \oplus A_2$ | 6 | 1,02 |
| $[-9] \oplus [1] \oplus A_2$ | 5 | 0,9 |
| $[-10] \oplus [1] \oplus A_2$ | 11 | 11 |
| $[-15] \oplus [1] \oplus A_2$ | 15 | 44 |
| $[-30] \oplus [1] \oplus A_2$ | 20 | 36,6 |

Table 1: Lattices of the forms $[k] \oplus A_3$ and $[k] \oplus [1] \oplus A_2$ for certain $k \leq 15$ and the lattice $U \oplus [36] \oplus [6]$. Time is given for the Intel Core i5 1.3GHz processor

performed by hand. We have also found a series of new reflective lattices. Some results yielded by the program are presented in the table. All lattices in this table, except $[-1] \oplus A_3$ and $[4] \oplus A_3$, are new. Moreover, we have proved the reflectivity of the lattices

$$[2] \oplus A_2 \oplus [1] \oplus \ldots \oplus [1] \quad \text{for} \quad n \leq 6$$

In the table, $U = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ denotes the standard two-dimensional hyperbolic lattice and $A_n$, the Euclidean root lattice of type $A_n$.

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