Vinberg's Algorithm

There is an affective algorithm of constructing the fundamental polyhedron P for hyperbolic reflection group. It works for each reflection group, but is efficient only for groups of the form $O_r(L)$. Choose any point $v_0 \in \mathbb{H}^n$; we shall call it the *basic point*. The fundamental domain P_0 of its stabilizer $O_r(L)_{v_0}$ is a polyhedral cone in $\mathbb{H}^n$. Let $H_1, \ldots, H_m$ be the faces of this cone and let $a_1, \ldots, a_m$ be the corresponding outer normals. Then we can define the half-spaces

$$H_k^- = \{x \in \mathbb{E}^{n,1} \mid (x, a_k) \leq 0\},$$

in addition, we can define in the same way the half-space H^- for every hyperplane H. Then we observe that the fundamental polyhedral cone is the intersection of the half-spaces of this cone: $P_0 = \cap_{k=1}^m H_k^-$

There exists the unique fundamental domain of the group $O_r(L)$ contained in P_0 and containing the point v_0. Its faces containing v_0 are formed by faces of the cone $H_1, \ldots, H_m$. The other faces $H_{m+1}, \ldots$ and the corresponding outward normals $a_{m+1}, \ldots$ are constructed by induction. Namely, for H_j we take a mirror such that the root a_j orthogonal to it satisfies the conditions

(1) $(a_j, v_0) < 0$;

(2) $(a_i, a_j) \leq 0$ for all $i < j$;

(3) the distance $\rho(v_0, H_j)$ is minimum subject to constraints (1) and (2).

The lengths of the roots a_j satisfy the following useful condition.

Proposition 1 *(Vinberg, 1984, [3]) The inner squares of roots in the quadratic lattice L are divisors of the doubled last invariant factor of L.*

Theorem 1 *(Vinberg, 1972, [1, 2]) The polyhedron P can be found in the following way:*

$$P = \bigcap_k H_k^-,$$

and, in addition, all H_k are the sides of P.

Thus, if the roots found at some step determine a polyhedron of finite volume, then Vinberg's algorithm terminates.

References

[1] E.B. Vinberg. The groups of units of certain quadratic forms. — Mat.Sb.(N.S.), 1972, 87 (129), p. 18 — 36.

[2] E.B. Vinberg. Some arithmetical discrete groups in Lobachevsky spaces. — In: Proc. Int. Coll. on Discrete Subgroups of Lie Groups and Appl. to Moduli (Bombay, January 1973). — Oxford: University Press, 1975, p. 323 — 348.

[3] E.B. Vinberg. Absence of crystallographic groups of reflections in Lobachevsky spaces of large dimension. — Trudy Moskov. Mat. Obshch., 1984, 47, p. 68 — 102.