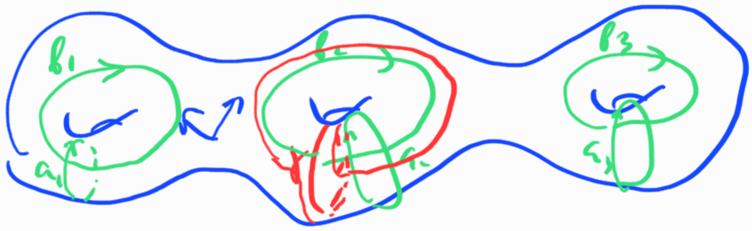
16.04.2021

Группа Торелли

S_g



$$Mod_g = \pi_0 \, \mathcal{D}iff^+(S_g)$$

$T_\gamma \in Mod_g$

$a_1 \; a_2 \; \cancel{b_2} \; a_2 \; a_3 \; b_3$

$T_g \subset Mod_g$

$Mod_g \curvearrowright H = H_1(S_g)$

$$\boxed{\begin{aligned} &T_\gamma a_1 = a_1 \\ &\vdots \\ &T_\gamma b_3 = b_3 \\ &T_\gamma b_2 = b_2 + a_2 \end{aligned}}$$

$-(b_1, a_1) = (a_1, b_1) = 1$

$(a_2, b_2) = 1$

$(a_3, b_3) = 1$

$$M = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

<u>Утв.</u> $Mod_g \curvearrowright H \quad (a, b) = (T \cdot a, T \cdot b)$

$$Mod_g \xrightarrow{p} Sp(2g)$$

<u>Утв.</u> p – эпиморфизм $\quad \det A = 1 \quad S^T = S$

$$Sp(2g) = \left\langle \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}, \right.$$

$$\left\{\left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \\ \hline -1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \end{array}\right)\right\}\rangle$$

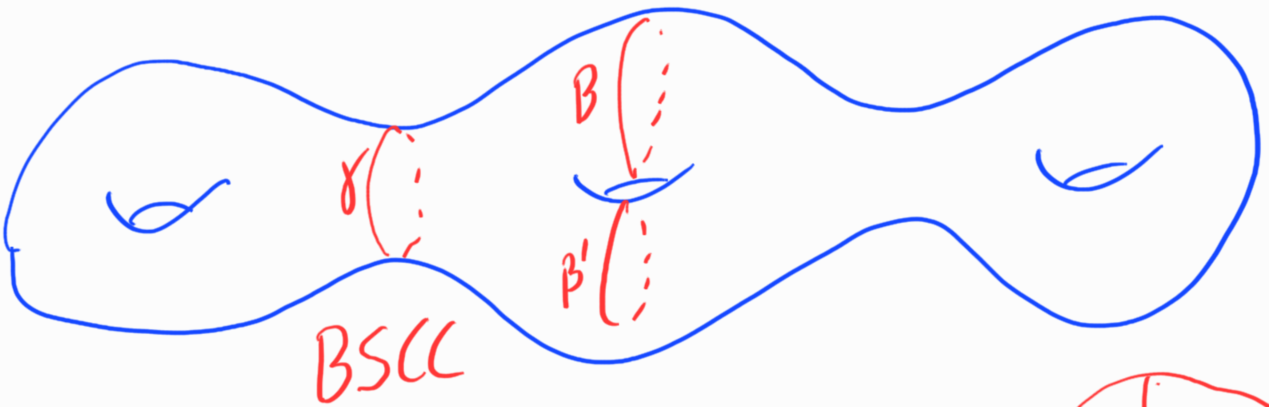
$$T_{a_i} : b_i \longmapsto b_i + a_i$$

$$T_{a_i + b_j} T_{b_j}^{-1} T_{a_i}^{-1} : \begin{array}{l} a_j \longmapsto a_j - a_i \\ b_i \longmapsto b_i + b_j \end{array}$$

$$0 \to \underset{\sim}{T_g} \to \mathrm{Mod}_g \to Sp(2g) \to 0$$

Группа Торелли

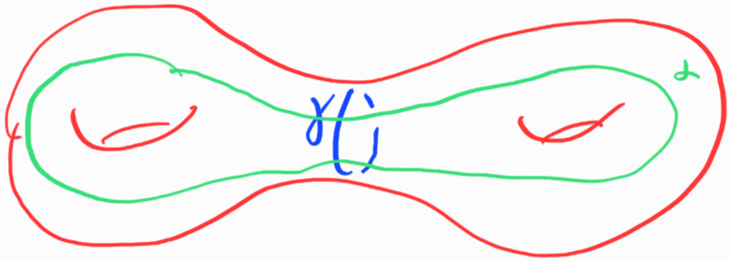
Powell '77



$$T_g = \langle \{ T_\gamma, \ T_\beta T_{\beta'}^{-1} \} \rangle$$

$$T_1 = 1$$

при $g \geq 3$ T_g конечно порождена.



Mod_g

$\Downarrow$

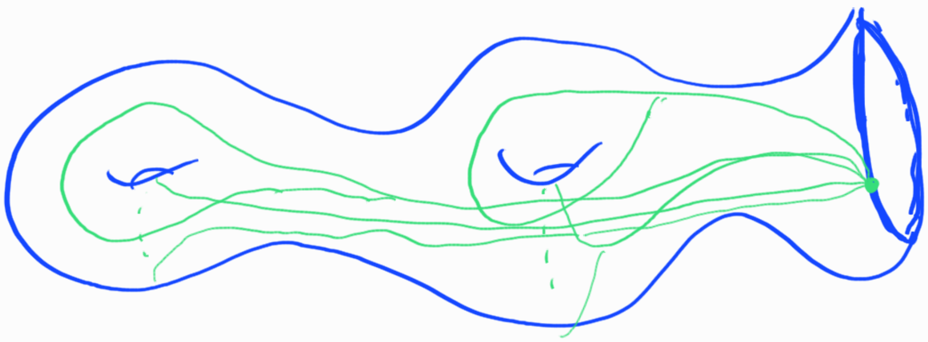
T_α

$\boxed{T_{\gamma'} = T_\alpha T_\gamma T_\alpha^{-1}}$ $\quad \underbrace{\gamma'} = \underbrace{T_\alpha \cdot \gamma}$

$(\mathcal{T}_g)_{ab} = \underline{\mathbb{Z}^\circ} \oplus \mathbb{Z}_2^\circ$

Гомоморфизм Джонсона

$S_{g,1} \quad \mathrm{Mod}_{g,1} > \mathcal{T}_{g,1}$



1) $\mathcal{T}_{g,1} \longrightarrow \mathrm{Hom}(H, H \wedge H)$

$T \in \mathcal{T}_{g,1}$

$\pi_1(S_{g,1}) = \langle a_1, b_1, \ldots, a_g, b_g \rangle$

$\alpha \in \pi_1(S_{g,1})$

$x_T(\alpha) = (T \cdot \alpha)\, \alpha^{-1} \in \pi_1(S_{g,1})$

$[x_T(\alpha)] \in \pi_1'$

$$[\pi_1,\pi_1]/[\pi_1,[\pi_1,\pi_1]] \cong H\wedge H$$

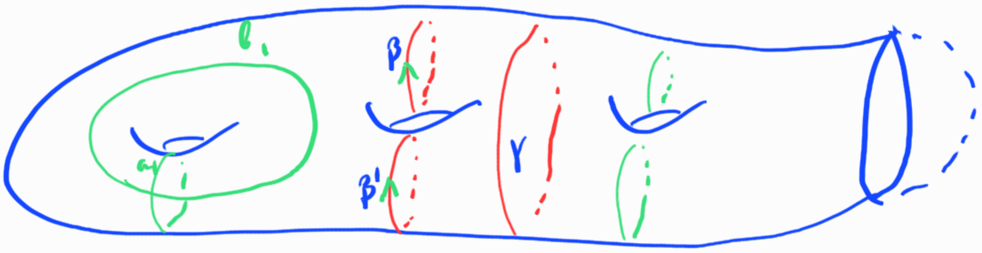
$$[a_i,b_j] \longmapsto [a_i]\wedge[b_j]$$

$$T \longmapsto \big([\alpha] \longmapsto [X_T(\alpha)]\big)$$

$$H \longrightarrow H\wedge H$$

2)
$$\begin{tikzcd} & \mathrm{Hom}(H, H\wedge H) \arrow[r] & H\otimes(H\wedge H) \\ \mathcal{T}_{g,1} \arrow[ur] \arrow[urr, bend right, "\tau"'] & & \end{tikzcd}$$

$$\underline{\operatorname{Im}\tau = H\wedge H\wedge H} = \Lambda^3 H$$



$$\underbrace{\textstyle\sum(a_i\wedge b_i)}\wedge[\beta]$$

$$\tau(T_\gamma)=0$$

$$\tau(T_\beta T_{\beta'}^{-1}) = \underbrace{(a_1\wedge b_1)\wedge \overset{\overset{-[\beta']}{\|}}{[\beta]}}$$

3) $\pi_1(U^1S_g) \longrightarrow \mathcal{T}_{g,1} \twoheadrightarrow \mathcal{T}_g$ $\quad \boxed{S_{g,1} \to S_g}$

$$H \xhookrightarrow{\wedge(\sum_i a_i\wedge b_i)} \Lambda^3H \longrightarrow \Lambda^3H/H$$

Утв. $T_g \to \Lambda^3 H / H$

$(T_g)_{abt} = \Lambda^3 H / H$

ядро Джонсона K_g

$$1 \to K_g \to T_g \to \Lambda^3 H / H \to 1$$

Утв. $K_g = \langle T_\gamma \mid \gamma - BSCC \rangle$

$\dim H_1 K_g^{\otimes\mathbb{C}} < \infty \quad g \geq 4$??

Гомология групп

$G \quad \mathbb{C}$

$\mathbb{C}\{[g_0 | g_1 | \ldots | g_n]\} = C_n(G)$

$\rho : G \to \mathbb{C}^\times$

$\partial : C_n(G) \to C_{n-1}(G)$

$[g_0 | g_1 | \ldots | g_n] \mapsto \rho(g_0)[g_1 | \ldots | g_n] -$

$- [g_0 g_1 | g_2 | \ldots | g_n] +$

$+ [g_0 | g_1 g_2 | \ldots | g_n] \ldots$

$\ldots + (-1)^{\circ} [g_0 | g_1 | \ldots | g_{n-1} g_n] +$

$+ (-1)^{\circ} [g_0 | \ldots | g_{n-1}]$

$\partial^2 = 0$ $\qquad H_*(G;\mathbb{C}_\rho)$

$\underline{K(G,1)}$ $\qquad H^*(G;\mathbb{C}_\rho)$

$V_n^k = \{\rho \in \{G_{ab} \to \mathbb{C}^\times\} \mid \dim H^k(G;\check{\mathbb{C}}_\rho) \geq n\}$

$\bigsqcup (\mathbb{C}^\times)^d$ $\qquad \ker[C^k \to C^{k+1}]$

$\mathbb{C}[G_{ab}]$ $\quad$ ACC

$R_n^k \longleftrightarrow V_n^k$

$R_1^1(G) \subseteq 0 \Leftrightarrow \dim \underbrace{H_1(G')}_{} < \infty$

T_g $\qquad \mathbb{C}G_{ab}$

$K_g \supset T_g'$

$\underline{V_1^1(G)}$ – конечно $\Longrightarrow H_1(G')$ – к/м

$\forall K \supset G'$

$H_1(K)$ – к/м

Ассоц. град. алг. Ли

$$G = G_1 \supset \underset{\substack{\| \\ G_2}}{[G,G]} \supset \underset{\substack{\| \\ G_3}}{[G,[G,G]]} \supset \dots$$

$$\bigoplus G_i / G_{i+1} = \underbrace{\mathrm{gr}_\bullet G}$$

$$\mathrm{gr}_i G \times \mathrm{gr}_j G \xrightarrow{[\cdot,\cdot]} \mathrm{gr}_{i+j} G$$

$$\mathrm{gr}_\bullet G \otimes \mathbb{C}$$

$$T_g \hookrightarrow \mathrm{Mod}_g \longrightarrow Sp(2g)$$

$$\underset{\substack{\| \\ \Lambda^3 H / H}}{(T_g)_{ab}} \curvearrowleft Sp(2g)$$

$$Sp(2g) \subset Sp_{\mathbb{C}}(2g)$$

$$Sp_{\mathbb{C}}(2g) \curvearrowright (T_g)_{ab} \otimes \mathbb{C} = \mathrm{gr}_1 T_g$$

$$Sp_{\mathbb{C}}(2g) \curvearrowright \mathrm{gr}_\bullet T_g \otimes \mathbb{C}$$

$$s[a,b]=[sa,sb]$$

$\mathrm{gr.}\, T_g \otimes \mathbb{C}$ $\quad$ $sp(2g)$-модули

$\Lambda^3 H/H$

$$t_i=\begin{array}{c|c} & i \\ i & \left(\begin{array}{c|c} \begin{matrix} 0 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \; 0 \end{matrix} & \\ \hline & \begin{matrix} 0 & & & \\ & \ddots & & \\ & & -1 & \\ & & & \ddots \; 0 \end{matrix} \end{array}\right) \end{array}$$

$$t_i \cdot a_i = a_i$$
$$t_i \cdot b_i = -b_i$$

$$[t_i, t_j]=0$$

$$h \subset sp(2g)$$

$$n^- \oplus h \oplus n^+ = sp(2g)$$

$$h^+ = \langle T_i, \underset{i<j}{S_{ij}}, F_{ij} \rangle$$

$$\begin{cases} T_i \cdot b_i = a_i \\ S_{ij}\cdot a_j = a_i \\ S_{ij} \cdot b_i = -b_j \\ F_{ij} \cdot b_i = a_i \\ F_{ij} \cdot b_j = a_j \end{cases} \quad i<j$$

$$[h, \underset{\underset{\mathfrak{g}_\alpha}{\cap}}{g}] = \alpha(h) g$$

$$\mathfrak{g} = sp(2g)$$

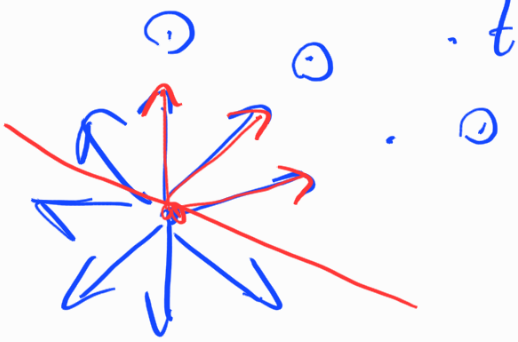
$$\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$$

$$\alpha \in \boxed{h^*}$$

$$\underset{i,j}{\pm t^i \pm t^j}; \pm 2 t^i$$

$\boxed{t^i - t^j,\ t^i + t^j,\ 2t^i \atop i<j}$ — полож. корни

$t_i < t_j \quad i<j$



$\mathfrak{g} \curvearrowright V$

$V = \bigoplus_\alpha V_\alpha$

$\alpha \in \mathbb{Z}\langle t^i \rangle \quad \in h^*$

$\lambda_i = t^1 + t^2 + \dots + t^i, \quad i \le g$

$\alpha = (\alpha_1 \dots \alpha_s)$

$\boxed{\alpha_1\lambda_1 + \dots + \alpha_s\lambda_s}$

$V(\alpha) = V(\alpha_1\lambda_1 + \dots + \alpha_s\lambda_s)$

$V(\lambda_1) = V$

$v_1 \wedge v_2 \longmapsto (v_1, v_2)$

$V(\lambda_2) = \mathrm{Ker}[V \wedge V \longrightarrow \mathbb{C}]$

$V(\lambda_3) = \mathrm{Ker}[V \wedge V \wedge V \longrightarrow V]$

$\boxed{V(2\lambda_2)}$ $\qquad v_1 \wedge v_2 \wedge v_3 \longmapsto (v_1 - v_2)v_3 + \sim$

$n = n_1 + \dots + n_k$

$V^{\otimes n} \supset W$

$\mathfrak{gl}(n)$

$sp(2n)$

$so(n)$

$$\Lambda^2 gr_1 T_g \xrightarrow[\beta]{[\,;\,]} gr_2 T_g$$

$\boxed{\text{Main}}$

$$\Lambda^2 V(\lambda_3) \cong \begin{cases} V(2\lambda_2) + V(0), & g = 3 \\ \vdots & g = 4, 5 \\ V(\lambda_6) + V(\lambda_4) + V(\lambda_4 + \lambda_2) + V(2\lambda_2) + V(\lambda_0) + V(\lambda_2) & g \geq 6 \end{cases}$$

$\underline{\text{T. Main}}$ $\quad \underline{\operatorname{Ker} \beta = V(2\lambda_2) + V(\lambda_0)}$

$$R_i^k(G) = \{\alpha \in H^1(G) \mid \dim H^k(H^*(G), u_\alpha) \geq i\}$$

$$u_\alpha = (\alpha \cdot) : H^n(G) \to H^{n+1}(G)$$

$$H^{k-1}(G) \xrightarrow{\alpha \cdot} H^k(G) \xrightarrow{\alpha \cdot} H^{k+1}(G)$$

$\underbrace{R_1^1(G)}_{V_1^1(G)} \ni \alpha \quad \exists \beta \neq c\alpha \quad \alpha\beta = 0$

$\beta \in H^1(G)$

$$\boxed{TG, \; U_1^1(G) \subset R_1^1(G)}$$

$$\underline{I.} \quad R_1^1(T_g) = \begin{cases} \{0\} & \text{при } g \geq 4 \\ H^1(T_g) & \text{при } g = 3 \end{cases}$$

$$\left(gr_1 T_g\right)^* \cong H^1 T_g$$

$$\begin{array}{c} gr_1 T_g \\ \| \wr \\ (T_g)_{ab} \\ \| \wr \\ H_1(T_g) \end{array}$$

Sullivan, Lambe

$H_1 T_g < \infty$

$$0 \to \left(gr_2 T_g\right)^* \xrightarrow{\beta^*} \left(\Lambda^2 gr_1 T_g\right)^*$$

$$\left(\Lambda^2 gr_1 T_g\right)^* = \Lambda^2 H^1 T_g \xrightarrow{\smile} H^2 T_g$$

$$a \smile b = 0$$

$$\Downarrow$$

$$a, b \in R_1^1(T_g)$$

$$\left(V(2\lambda_2) + V(0)\right)^* \xrightarrow{\sim} \left(\Lambda^2 V(\lambda_3)\right)^* \ni x \wedge y$$

$g = 3 \Rightarrow \Lambda^2 H^1 T_g \in \ker(\smile)$

$g \geq 4 \quad sp(2g) \qquad V(\lambda_3)^* = V(\lambda_3)$

$$V \to \Lambda^3 V \to V(\lambda_3)$$

$$\left(\sum_{i=1}^{g} a_i \wedge b_i\right) \wedge \cdot$$

$$V(\lambda_3) = \Lambda^3 V / V$$

$$x_1 \wedge x_2 \wedge x_3 \mapsto (x_1, x_2)\, x_3 + \ldots$$

$$[a_1 \wedge a_2 \wedge b_3] \subset V(\lambda_3)$$

$$v = [a_1 \wedge a_2 \wedge a_3] \in V(\lambda_3)$$

$$v \notin R_1^1(T_g)$$

$$v' \wedge v \notin \mathrm{Im} \quad V(2\lambda_2) + V(0)$$

$$R_1^1(T_g) \subset V(\lambda_3)$$ λ_3 – старший вес $V(\lambda_3)$

<u>Утв.</u> $R_1^1(T_g) \neq \{0\} \Rightarrow R_1^1(T_g) \ni v \in V(\lambda_3)_{\lambda_3} = \mathbb{C}\langle a_1 \wedge a_2 \wedge a_3 \rangle$

G – разр. группа Ли $(\mathfrak{h} \oplus \mathfrak{n}^+)$ $G \curvearrowright X$ – полное мн-е $(R_1^1(T_g))$ Тогда у G на X есть неподв. точка

$$h v = \underbrace{(t^1 + t^2 + t^3)(h)}_{=\lambda_3} v, \quad h \in \mathfrak{h}^*$$

$$t_1 v = \underbrace{t_1 a_1}_{=a_1} \wedge a_2 \wedge a_3 + a_1 \wedge \underbrace{\cancel{t_1 a_2}}_{=0} \wedge a_3 + a_1 \wedge a_2 \wedge \underbrace{\cancel{t_1 a_3}}_{=0} = v$$

$$t_2 v = v$$

$$t_3 v = v$$

$$t_k v = 0 \qquad k \geq 4$$

$V = \{a_1 \wedge a_2 \wedge a_3\}$

$\exists w : w \wedge v \in V(2\lambda_2) + V(0)$??