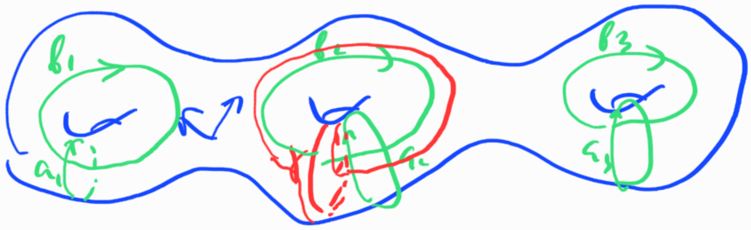
Группа Торелли

16.04.2021

S_g



$T_\gamma \in \mathrm{Mod}_g$

$a_1 \quad a_2 \quad \cancel{b_2} \quad a_2 \quad a_3 \quad b_3$

$$\mathrm{Mod}_g = \pi_0 \,\mathrm{Diff}^+(S_g)$$

$$T_g \subset \mathrm{Mod}_g$$

$$\mathrm{Mod}_g \curvearrowright H = H_1(S_g)$$

$$-(b_1, a_1) = (a_1, b_1) = 1$$

$$(a_2, b_2) = 1$$

$$(a_3, b_3) = 1$$

$$\boxed{\begin{aligned} T_\gamma a_1 &= a_1 \\ &\vdots \\ T_\gamma b_3 &= b_3 \\ T_\gamma b_2 &= b_2 + a_2 \end{aligned}}$$

$$M = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

Утв.

$$\mathrm{Mod}_g \curvearrowright H \qquad (a, b) = (T\cdot a, T\cdot b)$$

$$\mathrm{Mod}_g \xrightarrow{P} Sp(2g)$$

Утв. P – эпиморфизм $\quad \det A = 1 \quad S^T = S$

$$Sp(2g) = \left\langle \begin{pmatrix} A^T & 0 \\ 0 & A^{-1} \end{pmatrix}, \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}, \right.$$

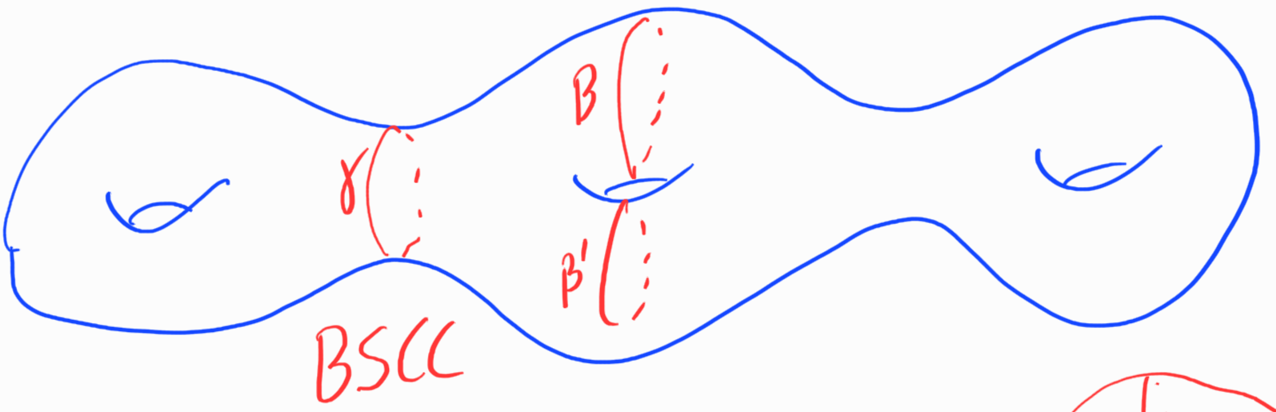
$$\left\langle \left\{ \left(\begin{array}{ccc|ccc} 0 & & & 1 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \\ \hline -1 & & & 0 & & \\ & 0 & & & 1 & \\ & & 0 & & & 1 \end{array} \right) \right\} \right\rangle$$

$$T_{a_i} : b_i \longmapsto b_i + a_i$$

$$T_{a_i + b_j} T_{b_j}^{-1} T_{a_i}^{-1} : \begin{array}{l} a_j \longmapsto a_j - a_i \\ b_i \longmapsto b_i + b_j \end{array}$$

$$0 \to \underbrace{T_g}_{\text{Группа Торелли}} \to \mathrm{Mod}_g \longrightarrow Sp(2g) \to 0$$

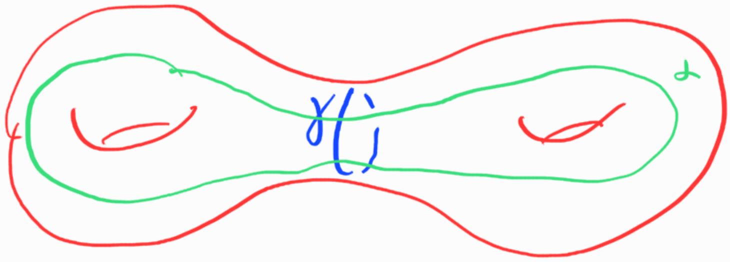
Powell '77



$$T_g = \left\langle \left\{ T_\gamma ,\ T_\beta T_{\beta'}^{-1} \right\} \right\rangle$$

$T_1 = 1$

при $g \geq 3$ T_g конечно порождена.



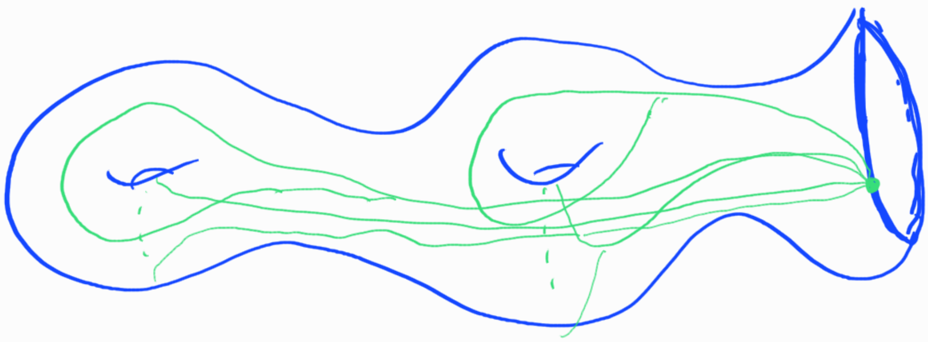
$$\mathrm{Mod}_g \ni T_\alpha$$

$$\boxed{T_{\gamma'} = T_\alpha T_\gamma T_\alpha^{-1}} \qquad \gamma' = T_\alpha \cdot \gamma$$

$$(T_g)_{ab} = \underline{\mathbb{Z}} \oplus \mathbb{Z}_2$$

Гомоморфизм Джонсона

$S_{g,1}$ $\quad \mathrm{Mod}_{g,1} \supset T_{g,1}$



1) $T_{g,1} \longrightarrow \mathrm{Hom}(H, H \wedge H)$

$T \in T_{g,1}$

$$\pi_1(S_{g,1}) = \langle a_1, b_1, \ldots, a_g, b_g \rangle$$

$\alpha \in \pi_1(S_{g,1})$

$$x_T(\alpha) = (T \cdot \alpha)\,\alpha^{-1} \in \pi_1(S_{g,1})$$

$$[x_T(\alpha)] \in \pi_1'$$

$$[\pi_1,\pi_1]/[\pi_1,[\pi_1,\pi_1]] \cong H\wedge H$$

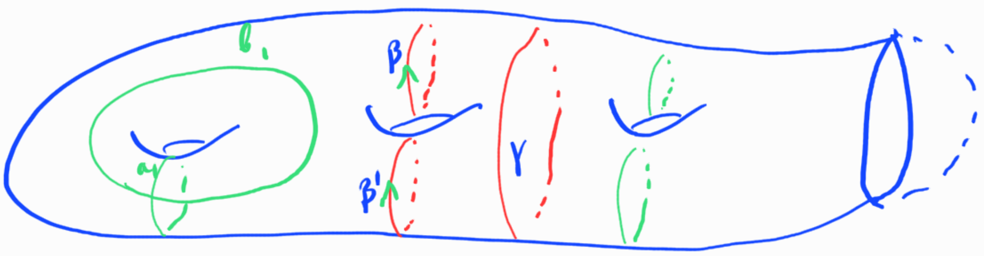
$$[a_i,b_j] \longmapsto [a_i]\wedge[b_j]$$

$$T \longmapsto \left([\alpha]\longmapsto [X_T(\alpha)]\right)$$

$$H \longrightarrow H\wedge H$$

2)
$$\begin{tikzcd}
& \mathrm{Hom}(H, H\wedge H) \arrow[r] & H\otimes(H\wedge H) \\
\mathcal{T}_{g,1} \arrow[ur] \arrow[urr, bend right, "\tau"'] & &
\end{tikzcd}$$

$$\underline{\mathrm{Im}\,\tau = H\wedge H\wedge H = \Lambda^3 H}$$



$$\underbrace{\sum(a_i\wedge b_i)}\wedge[\beta]$$

$$\tau(T_\gamma)=0$$

$$\tau(T_\beta T_{\beta'}^{-1}) = \underbrace{(a_1\wedge b_1)\wedge \overset{-[\beta']}{\overset{\parallel}{[\beta]}}}$$

3) $\pi_1(U'S_g) \longrightarrow \mathcal{T}_{g,1} \twoheadrightarrow \mathcal{T}_g$ $\quad \underline{\boxed{S_{g,1}\to S_g}}$

$$H \xhookrightarrow{\wedge(\sum_i a_i\wedge b_i)} \Lambda^3 H \longrightarrow \Lambda^3 H/H$$

Утв. $T_g \to \Lambda^3 H / H$

$(T_g)_{abt} = \Lambda^3 H / H$

ядро Джонсона K_g

$$1 \to K_g \to T_g \to \Lambda^3 H / H \to 1$$

Утв. $K_g = \langle T_\gamma \mid \gamma - BSCC \rangle$

$\dim H_1 K_g^{\otimes \mathbb{C}} < \infty \quad g \geq 4$??

Гомология групп

$G \quad \mathbb{C}$

$\mathbb{C}\{[g_0 | g_1 | \ldots | g_n]\} = C_n(G)$

$\rho : G \to \mathbb{C}^\times$

$\partial : C_n(G) \to C_{n-1}(G)$

$$[g_0 | g_1 | \ldots | g_n] \mapsto \rho(g_0)[g_1 | \ldots | g_n] - [g_0 g_1 | g_2 | \ldots | g_n] + [g_0 | g_1 g_2 | \ldots | g_n] \ldots + (-1)^{\circ}[g_0 | g_1 | \ldots | g_{n-1} g_n] + \ldots + (-1)^{\circ}[g_0 | \ldots | g_{n-1}]$$

$\partial^2 = 0$ $\quad H_*(G; \mathbb{C}_\rho)$

$\underline{K(G,1)}$ $\quad H^*(G; \mathbb{C}_\rho)$

$V_n^k = \{\rho \in \{G_{ab} \to \mathbb{C}^\times\} \mid \dim H^k(G; \mathbb{C}_\rho) \geq n\}$

$\bigsqcup (\mathbb{C}^\times)^d$

$R_n^k \longleftrightarrow V_n^k$

$R_1^1(G) \subseteq 0 \Leftrightarrow \dim \underline{H_1(G')} < \infty$

T_g $\quad \mathbb{C} G_{ab}$

$K_g \supset T_g'$

$\underline{V_1^1(G)}$ - конечно $\Longrightarrow$ $H_1(G')$ - к/м

$\forall K \supset G'$

$H_1(K)$ - к/м
