The Mostow Rigidity Theorem

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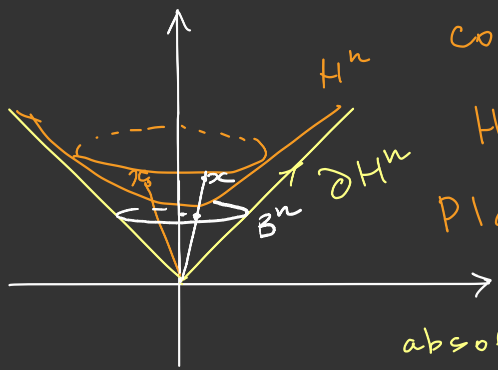
1) Hyperbolic (Lobachevsky) space $\mathbb{H}^n$

Let $\mathbb{E}^{n,1}$ be the Minkowski space, i.e. the $(n+1)$-dim real space equipped with the inner product $(x,y) = -x_0y_0 + x_1y_1 + \dots + x_ny_n$

Then $\boxed{\mathbb{H}^n := \{x \in \mathbb{E}^{n,1} \mid (x,x) = -1,\ x_0 > 0\}}$ | $I_{n,1} = \mathrm{diag}(-1,1,\dots,1)$



$\cosh \rho(x,y) = |(x,y)|$

$\mathrm{Isom}(\mathbb{H}^n) = PO_{n,1}(\mathbb{R}) = \{A \in GL_{n+1}(\mathbb{R}) \mid A^T I_{n,1} A = I_{n,1}\}/_{\pm I}$

Hyperplanes: $H_e = \{x \in \mathbb{H}^n \mid (x,e) = 0\}$

Planes are intersections of subspaces of $\mathbb{E}^{n,1}$ with $\mathbb{H}^n$

Isotropic vectors $((x,x)=0)$ are points on the absolute (or the boundary) $\partial \mathbb{H}^n$.

2) Convex polytopes and fundamental domains for discrete groups of isometries

Let $X^n = \mathbb{E}^n, \mathbb{S}^n$, or $\mathbb{H}^n$.

Def. A convex polytope: $P = \bigcap_{i \in I} H_i^-$, where $|I| < +\infty$, $\mathrm{int}(P) \neq \emptyset$

Def. A generalized convex polytope (or polyhedron) is

$P = \bigcap_{\alpha \in A} H_\alpha^-$, which is "locally" a usual convex polytope.

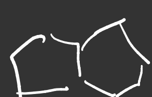
Def. A subgroup $\Gamma \subset \mathrm{Isom}(X^n)$ is called a discrete group of isometries/motions if all orbits Γx are discrete and all stabilizers Γ_x are finite.

Remark $\mathrm{Isom}(X^n)$ is a Lie group: $\mathrm{Isom}(\mathbb{S}^n) = O_{n+1}(\mathbb{R})$
$\mathrm{Isom}(\mathbb{E}^n) = \mathbb{R}^n \rtimes O_n(\mathbb{R})$
$\mathrm{Isom}(\mathbb{H}^n) = PO_{n,1}(\mathbb{R})$.

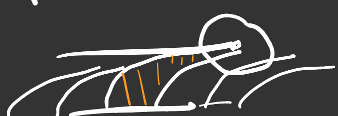
Thus, discrete isometry groups are discrete subgroups of Lie groups

Def. A (closed) subset $D \subset X^n$ is a fundamental domain for $\Gamma \subset \mathrm{Isom}(X^n)$ is

1) $\bigcup_{\gamma \in \Gamma} \gamma D = X^n$



$\Gamma = \mathbb{Z}^2 : \mathbb{R}^2$



2) $\mathrm{int}(\gamma D) \cap \mathrm{int}(\delta D) \neq \emptyset \iff \gamma = \delta$

3) $\forall p \in X^n\ \exists r > 0 : \#\{\gamma \mid B(p,r) \cap \gamma D \neq \emptyset\} < +\infty$

Def $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n) = PO_{n,1}$ is called a hyperbolic lattice if Γ has a finite volume fundamental domain D.
$(\mathrm{vol}\, D = \mathrm{vol}\, \mathbb{H}^n/\Gamma)$.
If D (or $\mathbb{H}^n/\Gamma$) is compact then Γ is called uniform lattice or cocompact.

Rem. Usually lattices in Lie groups are defined via Haar measures.
$\left(\underset{\text{lattice}}{\Gamma \subset G} \text{ is } \mathrm{vol}\,(G/\Gamma) < +\infty \right)$

Thm. Let $\Gamma \subset \mathrm{Isom}(X^n)$ be a discrete isometry gp. Then Γ has a fundamental domain which is a generalized convex polytope. (the Dirichlet domain)

Definition

Гиперболическое многообразие: $M = \mathbb{H}^n/\Gamma$, где $\Gamma < \mathrm{Isom}(\mathbb{H}^n)$ torsion-free

M компактно $\Leftrightarrow$ Γ - кокомпактная решётка (uniform hyperbolic lattice) $\Leftrightarrow$ фунд. мн-к D компактен

$\mathrm{vol}(M) < +\infty \Leftrightarrow \Gamma$ is a hyperbolic lattice $\Leftrightarrow \mathrm{vol}(D) < +\infty$.

Main goal: The Mostow Rigidity Theorem

$\boxed{n \geq 3}$!!

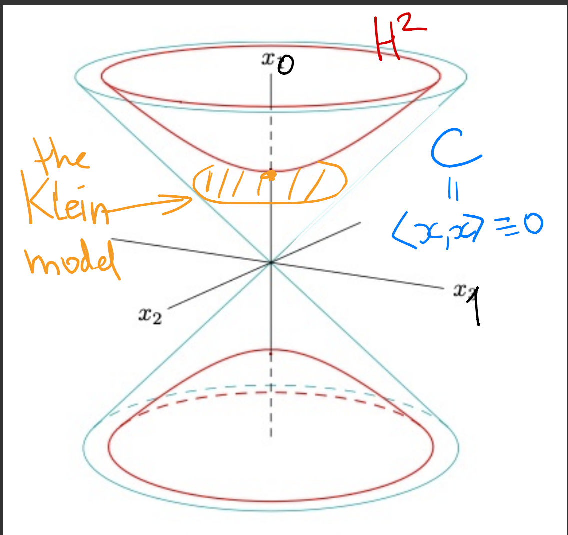
Compact hyperbolic n-manifolds $M_1 = \mathbb{H}^n/\Gamma_1$ and $M_2 = \mathbb{H}^n/\Gamma_2$ are homeo

$\Leftrightarrow$ $\Gamma_1 \simeq \Gamma_2$ as abstract groups

$\Leftrightarrow$ $\exists g \in PO_{n,1}(\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^n)$: $g \Gamma_1 g^{-1} = \Gamma_2$

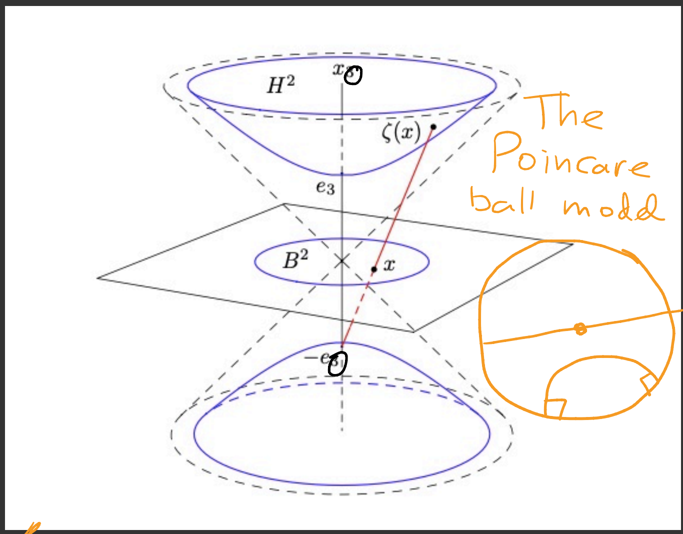
$\Leftrightarrow$ M_1 and M_2 are isometric.

3) Some hyperbolic geometry and examples of hyperbolic manifolds.



the Klein model

C = $\langle x, x\rangle = 0$



The Poincaré ball model

$\partial H^n \simeq S^{n-1}$

$\overline{H}^n = H^n \cup \partial H^n$

$\overline{H}^n \simeq$ closed ball in $\mathbb{R}^n$.

Upper half-space:

composition of an inversion and a reflection.

$H^2 = \{ z = a + bi \mid b > 0 \}$

$\partial H^2 = \{ b = 0 \} \cup \{\infty\} \simeq S^1$

$H^3 = \{ (z, t) \mid t > 0 \}$

$\partial H^3 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

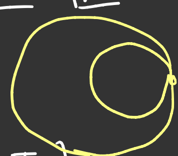
Theorem.

1) $\mathrm{Isom}(H^n) \simeq PO_{n,1}(\mathbb{R}) = O_{n,1}(\mathbb{R}) / \{\pm I\}$

2) $H^n \simeq PO_{n,1}(\mathbb{R}) / O_n(\mathbb{R})$.

3) The hyperbolic sphere $\{ x \in H^n \mid \rho(x, p) = \text{const} \} \overset{\text{isom}}{\simeq} S^{n-1}$ with a center $p \in H^n$

4) the horosphere $\{ x \in H^n \mid \underline{\langle x, p\rangle = \text{const}} \} \overset{\text{isom}}{\simeq} E^{n-1}$ with a center $p \in \partial H^n$

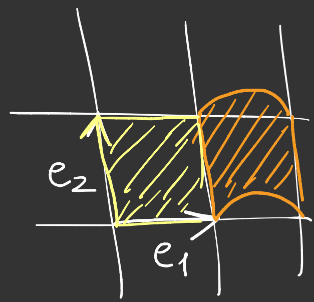


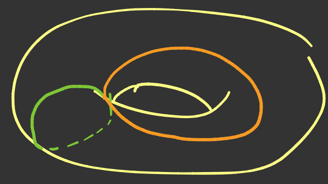
5) $\mathrm{Isom}^+(H^2) \simeq PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \{\pm I\}$

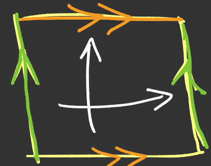
6) $\mathrm{Isom}^+(H^3) \simeq PSL_2(\mathbb{C}) = SL_2(\mathbb{C}) / \{\pm I\}$.

Examples

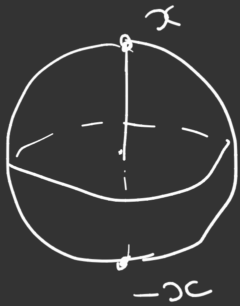
1) $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$

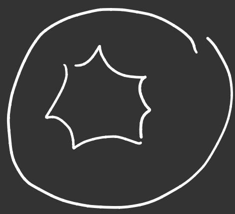




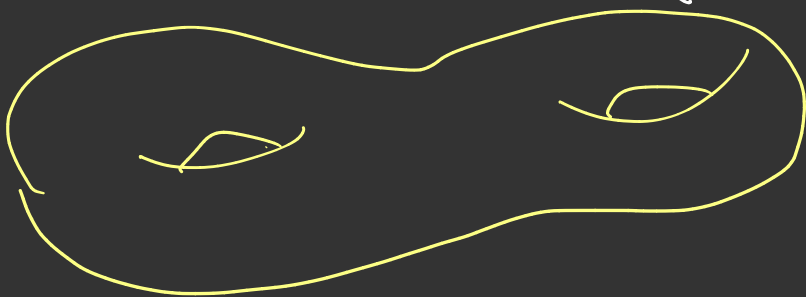


2) $\mathbb{RP}^2 \simeq S^2/\mathbb{Z}_2$





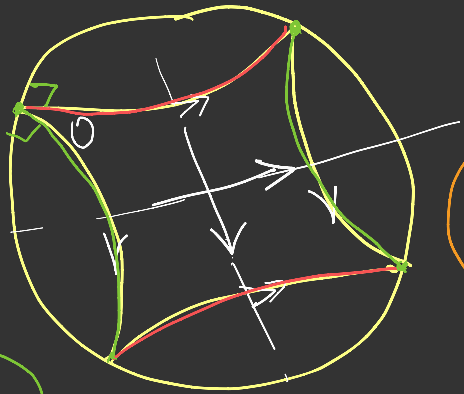
3) $S_g \simeq H^2/\Gamma$ if $g \geq 2$.

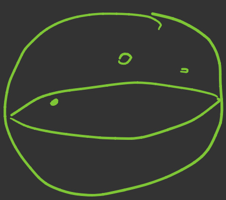


(Gauss-Bonnet)

$$\int_{S_g} \underset{\wedge\, 0}{K}\, dA = 2\pi \underset{\wedge\, 0}{\chi(S_g)}$$

4) Finite-volume hyperbolic surface with one cusp:





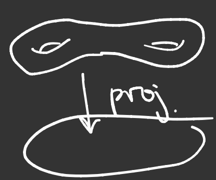
$\simeq$ homeo

$\cong$ $H^2/\mathbb{Z}*\mathbb{Z}$

An orbifold also has charts of the form $\mathbb{R}^n/\text{finite group}$ or $S^n/\text{finite grp}$ or $H^n/\text{finite grp}$

The Selberg Lemma: Any finitely generated linear group has a finite index torsion free subgrp.

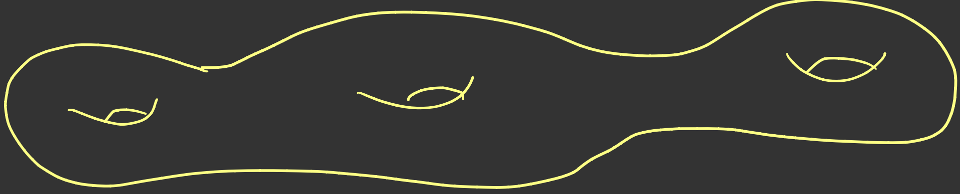
Corollary Let $\mathcal{O} = H^n/\Gamma$ is an orbifold, then $\exists$

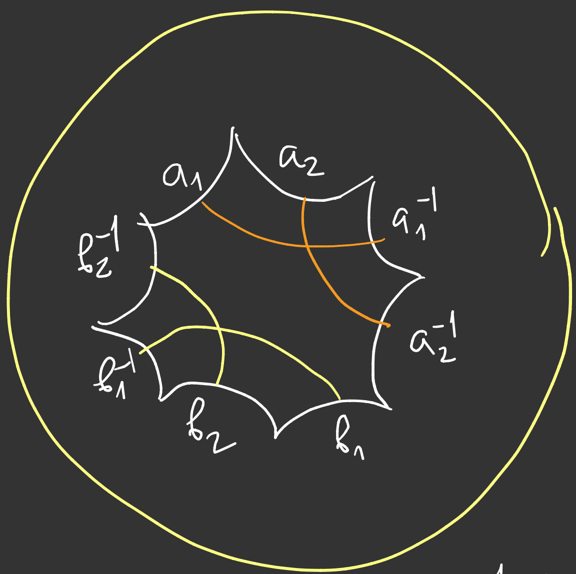


$M = H^n/\tilde{\Gamma}$ is a finite sheeted cover.

$\mathcal{O} = H^n/\Gamma$

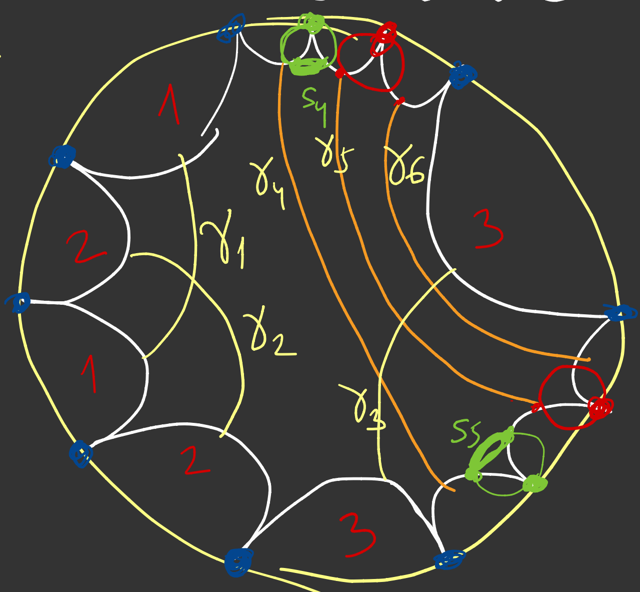
I $S_g = T^2 \# \dots \# T^2$



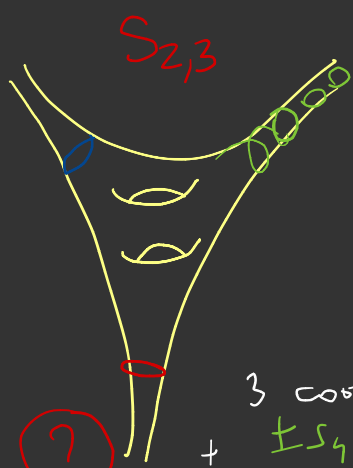


$\simeq$

II $3e^0 \cup 6e^1 \cup e^2$ $\quad 1-6+3=-2$



homeo $\simeq$



$\dim_{\mathbb{R}} M(S_{2,3}) = 15$?

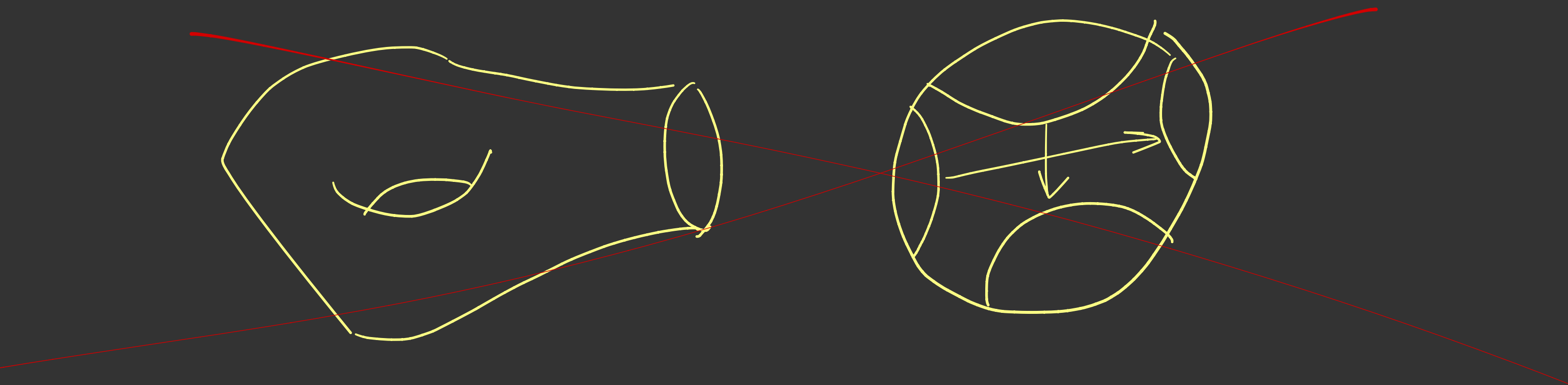
+ 3 соотн-я: $\pm s_4 = \pm s_5$ ...

$\dim_{\mathbb{R}} M = 6g-6$ (без каспов)

$+ 6$ для 3х каспов

$\begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}$ – гиперб. сдвиг в $PSL_2(\mathbb{R})$

$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ в $\mathbb{R}^{2,1}$



Due to Sasha Kolpakov: