Research Seminar Fall 2020

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1. Introduction. Arithmetic and quasi-arithmetic hyperbolic reflection groups.

(1) Exercises/Problems on hyperbolic reflection groups

(1) Let $\Gamma(m)$ be a Coxeter group given by a diagram: $\circ \overset{\infty}{=\!=} \circ \overset{m}{\text{---}} \circ$

(a) Check that $\Gamma(m)$ is a hyperbolic group, i.e. $\Gamma(m) \curvearrowright \mathbb{H}^2$ for all $m \geq 2$.

(b) Draw the fundamental domain $P(m)$ for $\Gamma(m)$. Is it compact or not? Find $\mathrm{vol}(P(m))$.

(c) Find all $m \geq 2$, s.t. $\Gamma(m)$ is arithmetic/quasi-arithm. and determine the ground field $k(\Gamma)$.

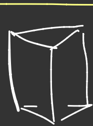
(2)** Let $\Gamma(m)$ be a Coxeter gp, given by $\circ\text{---}\circ\overset{m}{\text{---}}\circ$

(a) The same question — " — as in 1(a)

(b) — " —

(c) — " —

(3) Find all compact Coxeter simplices in $\mathbb{H}^n$.

(4) A prism in $\mathbb{H}^n$ is a polytope, which is combinatorially a direct product of a simplex and a closed interval. E.g. a 3-prism  $= \Delta \times I$
triangle closed interval
(we don't assume the bases parallel to each other.

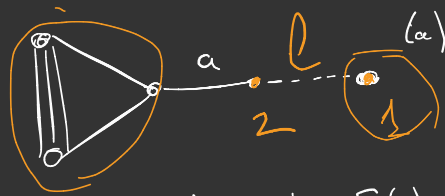
(a) Prove that the upper and the lower bases of any acute-angled prism in $\mathbb{H}^n$ can not be parallel to each other, i.e. they are divergent.

(b) Prove that any Coxeter, or acute-angled prism in $\mathbb{H}^n$ can be cut in 2 prisms with one common base, orthogonal to its adjacent facets.

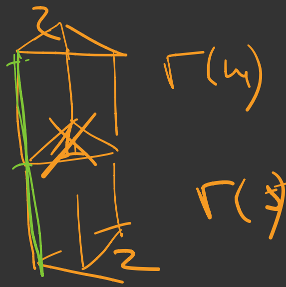
(c) Prove that there are no compact Coxeter prisms in $\mathbb{H}^{\geq 6}$.

⑤ Consider the prism:

$\Gamma(a)$:



(a) Check that $\Gamma(a)$ is a compact prism.



(b) Find all a, such that $\Gamma(a)$ is arithmetic and properly quasi-arithmetic. Determine the ground field k.

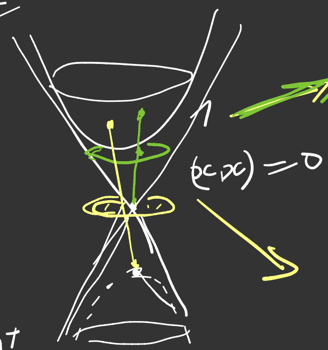
(c) Glue $\Gamma(4)$ and $\Gamma(5)$ together by their common triangular base. Let Γ be this new prism. Is it arithm, quasi-arith or neither?

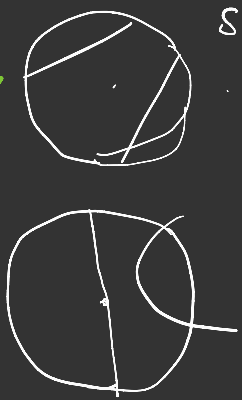
$\underline{\det G(\Gamma(a)) = 0}$

$\mathbb{H}^n \subset E^{n,1}$

$\cosh \rho(x,y) = -|\langle x,y\rangle|$

$-x_0y_0 + x_1y_1 + \ldots + x_ny_n$





$S^{n-1} \simeq \partial H^n$

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inversion + reflection

Thm (1) Let $p \in H^n$, then $\{x \mid \rho(p,x) = \text{const}\} \simeq S^{n-1}$

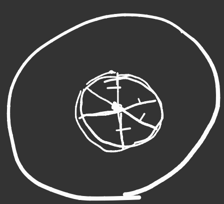
(2) If $p \in \partial H^n$, then $\{\underline{x \in H^n} \mid \langle x,p\rangle = \text{const}\} \simeq E^{n-1}$

(horospheres)

орисфера

Proof:

1)



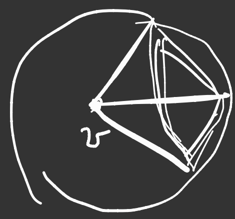
2)

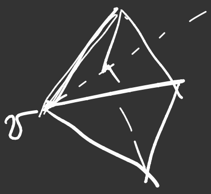
$\simeq E^{n-1}$

$$\begin{cases} \langle x,p\rangle = \langle y,p\rangle = \text{const} \\ (p,p) = 0 \end{cases}$$

$\langle x,y\rangle$

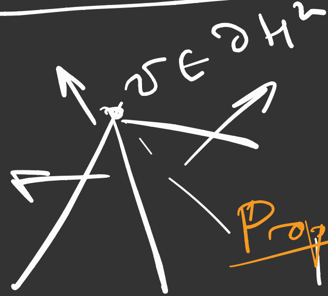
Thm If P is a compact acute-angled ($\le \frac{\pi}{2}$) polytope in $\mathbb{H}^n$, then it is simple.





elliptic Coxeter diagrams $\Leftrightarrow$ finite reflection gps on S^n

parabolic $\Leftrightarrow$ refl gps on E^n



$e_1, \dots, e_{n+1}$ vectors

linearly dependent!

Prop 1. If P is non-compact quasi-arith
$\Rightarrow k(P) = \mathbb{Q}$

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if $[k:\mathbb{Q}] > 1 \Rightarrow \exists\, \sigma \neq \mathrm{id} : k \to \mathbb{R}$

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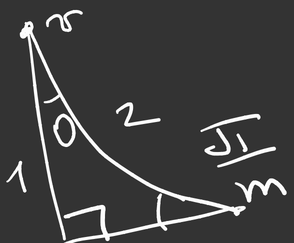
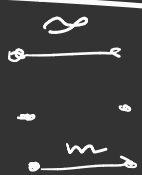
(1)





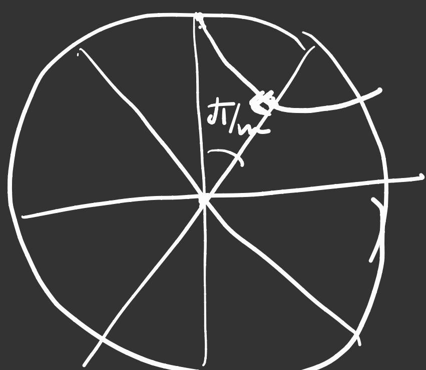
parabolic

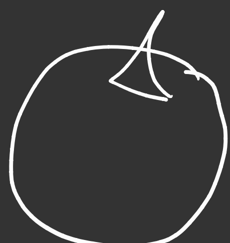
elliptic

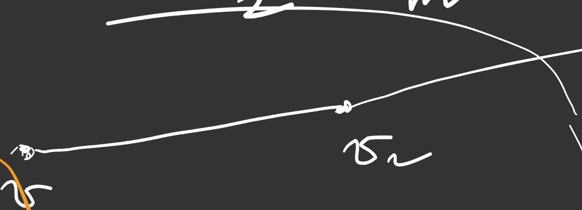
$$S_\Delta = \pi - (\alpha + \beta + \gamma)$$

$$= \pi - \frac{\pi}{2} - \frac{\pi}{m}$$

$$= \frac{\pi}{2} - \frac{\pi}{m}$$







$m = 2, 3, 4, 6.$

$k(\sqrt{\Gamma_m}) = \mathbb{Q}$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & -\cos\frac{\pi}{m} \\ 0 & -\cos\frac{\pi}{m} & 1 \end{pmatrix}$$

$\cos^2 \frac{\pi}{n} \in \mathbb{Q}$

② General exposition: discrete subgps of Lie groups.

Let G be a (semi-) simple Lie group (e.g. $G = PO(n,1) = \mathrm{Isom}(\mathbb{H}^n)$).

Def $\Gamma \underset{\text{lattice}}{\overset{\text{subgp}}{\subset}} G$ if $\mathrm{vol}(G/\Gamma) < +\infty$.

$$\pi : G \to G/\Gamma$$

$\uparrow$ μ Haar measure

Def. Γ is uniform if G/Γ is compact.

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k is an algebraic (totally real) number field

Let $\widetilde{G}$ be an admissible simple k-group (for G), i.e.

1) $\widetilde{G}(\mathbb{R}) \cong G$
2) $\widetilde{G}^{\sigma}(\mathbb{R})$ is compact for $\forall\, \sigma : k \to \mathbb{R}$, $\sigma \neq \mathrm{id}$.

Thm If Γ is commensurable with $\widetilde{G}(\mathcal{O}_k)$ then Γ is a lattice.

(arithmetic lattices).

1) General picture, examples.

2) Hyperbolic reflection gps. (geometry & combinatorics) Stepan

3) Quasi-arithmetic, arithmetic groups

3.1) Number theory M. Mzalla

3.2) fields; quadratic forms, quadratic lattices O_k^d

3.3) Quasi-arithmetic reflection gps Khusrav

4) Hyperbolic orbifolds and manifolds

Open problems

1) Vinberg '2012 Examples of compact quasi-arithmetic gps in H^2, preserving some isotropic q.f. Is it possible to do the same in $H^{\geq 3}$?

2) Some open questions in paper B-K.