Linear Algebra

Lecture 1: Preliminaries

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Operations

By an operation on a set M we mean a map

$$\circ \colon M \times M \to M$$

Examples of algebraic structures $(M, \circ)$:

- $(\mathbb{N}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$
- $(\mathbb{N}, \cdot)$, $(\mathbb{Z}, \cdot)$, $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$, $(\mathbb{C}, \cdot)$
- A set $C[0,1]$ of continuous functions on $[0,1]$ with an operation of composition: $(f \circ g)(x) = f(g(x))$

Polynomials

Suppose $\Bbbk$ is some set of numbers.

Then a polynomial over $\Bbbk$ is:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

where $a_n \neq 0$. Then

$\deg(p) := n$;

$\Bbbk[x] :=$ is a set of all polynomials;

$\Bbbk[x]_n := \{p \in \Bbbk[x] \mid \deg(p) \leq n\}$.

Matrices

Suppose $\Bbbk$ is some set of numbers.

A matrix over $\Bbbk$ is a table $m \times n$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$\mathrm{Mat}_{m,n}(\Bbbk) :=$ a set of all matrices;

$\mathrm{Mat}_{n}(\Bbbk) := \mathrm{Mat}_{n,n}(\Bbbk)$;

$\mathrm{GL}_n(\Bbbk) = \mathrm{GL}(n, \Bbbk) :=$

$:= \{A \in \mathrm{Mat}_n(\Bbbk) \mid \det(A) \neq 0\}$.

Addition of matrices

For $A, B \in \mathrm{Mat}_{m,n}(\mathbb{k})$ we define $A + B \in \mathrm{Mat}_{m,n}(\mathbb{k})$ as:

$$A + B =$$

$$= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn}; \end{bmatrix}$$

Multiplication of matrices

For $A \in \mathrm{Mat}_{m,n}(\Bbbk)$ and $B \in \mathrm{Mat}_{n,r}(\Bbbk)$ we define $A \cdot B \in \mathrm{Mat}_{m,r}(\Bbbk)$ as:

$$A \cdot B = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & \dots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nr} \end{bmatrix} =$$

$$= \left[c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \right]_{i=1,j=1}^{m,r}$$

c_{ij} is $\langle (a_{i1}, a_{i2}, \dots, a_{in}), (b_{1j}, b_{2j}, \dots, b_{nj}) \rangle$

Exercises

Are the following pairs $(M, \circ)$ correctly defined algebraic structures:

- $(\mathbb{Z}_{-}, \cdot)$, where $\mathbb{Z}_{-} = -\mathbb{N} = \{z \in \mathbb{Z} \mid z < 0\}$
- $(\Bbbk[x], +)$, $(\Bbbk[x], \cdot)$
- $(\Bbbk[x]_n, +)$, $(\Bbbk[x]_n, \cdot)$
- $(\mathrm{Mat}_{m,n}(\Bbbk), +)$, $(\mathrm{Mat}_{m,n}(\Bbbk), \cdot)$
- $(\mathrm{Mat}_n(\Bbbk), \cdot)$
- $(\mathrm{GL}_n(\Bbbk), +)$, $(\mathrm{GL}_n(\Bbbk), \cdot)$?

Isomorphism of algebraic structures

Algebraic structures $(M, \circ)$ and $(N, *)$ are isomorphic if there exists a bijective map $f\colon M \to N$, s.t. $f(a \circ b) = f(a) * f(b)$.

We denote it as $(M, \circ) \simeq (N, *)$.

Example: A map $a \mapsto 2^a$ defines an isomorphism $(\mathbb{R}, +) \simeq (\mathbb{R}_+, \cdot)$.

Groups

A set G with an operation $\circ$ is called a group if it has the properties:

- $(a \circ b) \circ c = a \circ (b \circ c)$ (associativity)
- there exists (the identity) $e \in G$, such that $a \circ e = e \circ a = a$ for all $a \in G$
- for any $a \in G$ there $\exists a^{-1} \in G$ (an inverse), s.t. $a \circ a^{-1} = a^{-1} \circ a = e$.

A group G is Abelian if $a \circ b = b \circ a$ (commutativity).

Examples of groups

- $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$
- $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$, $(\mathbb{C}, \cdot)$
- $(\mathbb{k}[x], +)$, $(\mathbb{k}[x]_n, +)$
- $(\mathrm{Mat}_{m,n}(\mathbb{k}), +)$
- $(\mathrm{GL}_n(\mathbb{k}), \cdot)$
- $(C[0, 1], +)$.

Exercise: Verify it!

Rings

A set K with two operation $+$ and $\cdot$ is called a ring if it has the properties:

- $(K, +)$ is Abelian group (the additive group)
- $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c, \in K$ (distributive laws).

Exercises

- $a0 = 0a = a$ for any $a \in K$
- $a(-b) = (-a)b = -ab$ for any $a, b \in K$
- $a(b - c) = ab - ac$ for any $a, b, c \in K$
- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are commutative associative rings with unities
- $2\mathbb{Z}$ is commutative associative ring without unity

Fields

A field is a commutative associative ring with unity where every nonzero element is invertible.

- Usually denoted by $\mathbb{k}$ or $\mathbb{F}$
- A ring $\{0\}$ is not regarded as a field
- $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields (Verify!)
- $\mathbb{Z}_2 = \{0, 1\}$ can be considered as a field (Verify!).

Vector Spaces

A set V with operations of addition $+ \colon V \times V \to V$ and scalar multiplication $\cdot \colon \Bbbk \times V \to V$ is a vector space over $\Bbbk$, if for all $v, v_1, v_2, v_3 \in V$ and $\lambda, \mu \in \Bbbk$

- $(V, +)$ is Abelian group and
- $(\lambda\mu)v = \lambda(\mu v)$
- $(\lambda + \mu)v = \lambda v + \mu v$
- $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$
- $1 \cdot v = v.$

Exercises/Examples

- $0 \cdot v = v$ and $(-1)v = -v$ for any $v \in V$
- $V = 0$, $V = \mathbb{k}$ are vector spaces
- $V = \mathbb{k}^n = \{(x_1, x_2, \dots, x_n) \mid x_j \in \mathbb{k}\}$ is a vector space, where $\lambda(x_1, x_2, \dots, x_n) := (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$ and $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) =$ $= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$
- $(\mathrm{Mat}_n(\mathbb{k}), +, \cdot)$ is a vector space.