Linear Algebra

Lecture 3: Convex Sets and Motions

Nikolay V. Bogachev

Moscow Institute of Physics and Technology,
Department of Discrete Mathematics,
Laboratory of Advanced Combinatorics and Network Applicationss

Coordinate and Matrix Form of Affine Transformation

Suppose $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ is an affine transformation. Then $df\colon \mathbb{k}^2 \to \mathbb{k}^2$ is a linear map. In the vectorization form, $f\colon X \mapsto df(X) + B$.

That is,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Convex Sets

Suppose $\mathbb{A}$ is an affine space.

$AB = [A, B] = \{\lambda A + (1 - \lambda)B \mid 0 \leq \lambda \leq 1\}$ is a segment.

$M \subset \mathbb{A}$ is convex if with any points $A, B \in M$ it contains the whole AB.

Planes are convex sets. If M_1, M_2 are convex, then $M_1 \cap M_2$ is convex.

Convex Hull

A convex linear combination of points in $\mathbb{A}$ is their barycentric combination with non-negative coefficients.

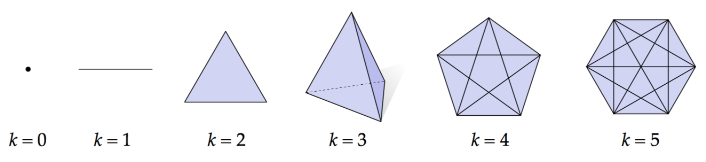
For any $A_0, A_1, \ldots, A_k \in M$, where M is convex, M also contains every convex combination $\sum \lambda_j A_j$.

For any $M \subset \mathbb{A}$, the set $\operatorname{conv}(M)$ of all convex combinations of points in M is convex: $\operatorname{conv}(M)$ is a convex hull of M.

Simplex

A convex hull of a system of affinely independent points $A_0, A_1, \dots, A_k \in \mathbb{A}$ is a $k\text{-dim}$ simplex (or k-simplex).

That is, 0-simplex is a point, 1-simplex is a segment, 2-simplex is a triangle, etc.



Motions/Isometries of Euclidean Space

Suppose $\mathbb{A} = \mathbb{E}^n$, Then

$$\mathrm{Isom}(\mathbb{E}^n) = \{f \in \mathrm{Aff}(\mathbb{E}^n) \mid \forall X, Y \in \mathbb{E}^n \\ \rho(f(X), f(Y)) = \rho(X, Y)\}.$$

is the isometry group of $\mathbb{E}^n$.

The stabilizer of some point is a subgroup of $\mathrm{GL}(n, \mathbb{R})$ that preserves the standard inner product: it is $\mathrm{O}(n, \mathbb{R})$.

Reflections

Suppose $H \subset \mathbb{E}^n$ is a hyperplane. That is, $H = \{x \in \mathbb{R}^n \mid (x, e) + t = 0, \|e\| = 1, t \in \mathbb{R}\}$.

Then an orthogonal reflection $\mathcal{R}_{e,t} = \mathcal{R}_H$ with respect to $H_{e,t} := H$ is

$$\mathcal{R}_{e,t}(x) = x - 2((e, x) + t)e.$$

$\mathrm{Isom}(\mathbb{E}^n)$ is generated by reflections.

Semidirect Product

We say that G is decomposed into the semidirect product of its subgroups N and H if

- N is a normal subgroup
- $N \cap H = \{e\}$
- $G = NH$.

We denote it by $G = N \rtimes H$.

Semidirect Product

- $S_n = A_n \rtimes \langle (12) \rangle$
- $S_4 = V_4 \rtimes S_3$
- $\mathrm{GL}(n, \mathbb{k}) = \mathrm{SL}(n, \mathbb{k}) \rtimes \{diag(\lambda, 1, \dots, 1) \mid \lambda \in \mathbb{k}^*\}$
- $\mathrm{Aff}(\mathbb{A}) = T(\mathbb{A}) \rtimes \mathrm{GL}(V)$
- $\mathrm{Isom}(\mathbb{E}^n) = T(\mathbb{E}^n) \rtimes \mathrm{O}(n, \mathbb{R})$