Linear Algebra

Lecture 6: Convex Polyhedra II

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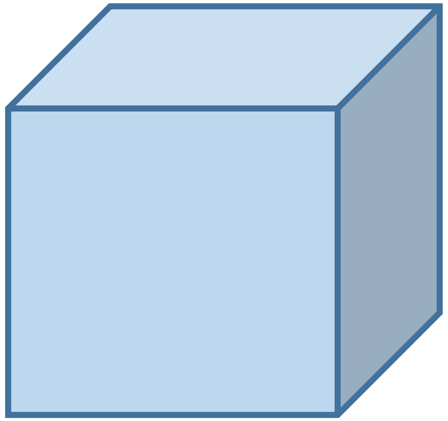
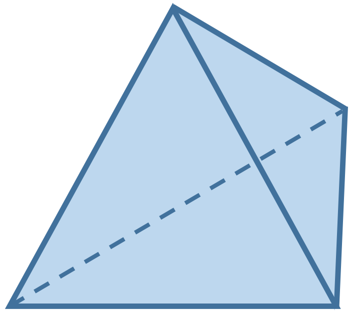
Convex Polyhedron

A convex polyhedron (or a convex polytope) is an intersection of finitely many half-spaces (sometimes, nonempty interior is required).



Parallelepiped

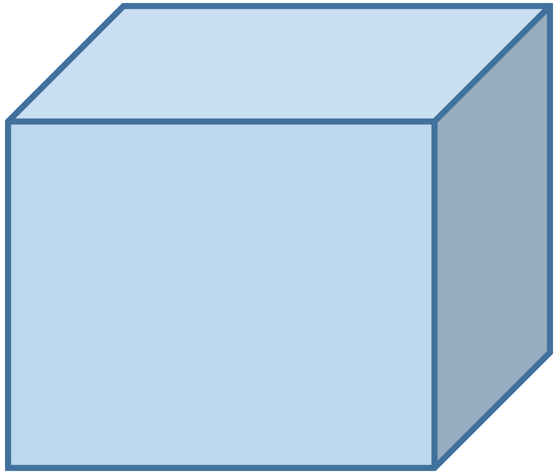


Simplex

Minkowski-Weyl Theorem

M is a convex polyhedron iff M is a convex hull of finitely many points.

$$M = \text{conv}\,\{\text{vertices of } M\}\ ?$$



Faces of Polyhedra

A face of a convex polyhedron M is a nonempty intersection of M with some of its supporting hyperplanes.

- A 0-dim face is called a vertex
- A 1-dim face, an edge
- A 2-dim face, a plane
- An $(n-1)$-dim face, a hyperface or a facet

Faces of Polyhedra

Every face F of M is of the form

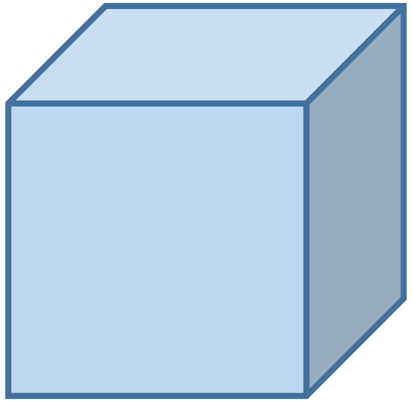
$$F = M \cap \left(\bigcap_{j \in J} H_{f_j} \right), \text{ where } J \subset \{1, \dots, m\}$$

Since a convex polyhedron is determined by a system of linear inequalities, its faces can be obtained by replacing some of these inequalities with equalities.

Example

A parallelepiped $\{x \mid 0 \le x_j \le 1, \ \forall j\}$ has the faces obtained by setting some of x_k to 0 or 1.

Its vertices are points $\{\, x \mid x_j = 0 \text{ or } 1, \ \forall j \,\}$.



(1,1,1)

$\{\, (1, x_2, x_3) \mid 0 \le x_j \le 1, \ j = 2,3 \,\}$

(1,0,0)

Vertices as Extreme Points

The extreme points of a convex polyhedron M are exactly its vertices.

Proof: If a point $X \in \partial M$ is an interior point of an interval in M, then a supporting hyperplane through X contains this interval. Hence X is not a vertex of M.

Conversely, if X is not a vertex of M, then $X \in \mathrm{int}(F)$ of $\dim F > 0$, i.e. is not extreme.

Linear Programming

The maximum of an affine-linear function on a bounded convex polyhedron M is attained at a vertex.

Proof: Every $X \in M$ is of the form:

$$X = \sum_{j=1}^{k} \lambda_j A_j, \ \sum_{j=1}^{k} \lambda_j = 1, \ \ \lambda_j \geq 0.$$

Then $f(X) = \sum_{j=1}^{k} \lambda_j f(A_j) \leq \max_j f(A_j)$.

The Maximum Profit Problem

A company processes resources $R_1, \dots, R_m$ of amounts $b_1, \dots, b_m$, respectively, and wants to produce products $P_1, \dots, P_n$ of amounts $x_1, \dots, x_n$, respectively.

Let a_{ij} be the amount of R_i needed to produce a unit of P_j. Clearly, the following inequalities should hold:

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad x_j \ge 0, \quad i = 1, \dots, m$$

The Maximum Profit Problem

They determine the convex polyhedron M in the n-space with coordinates $x_1, \dots, x_n$.

To maximize the profit, one needs to find the point $x_1, \dots, x_n \in M$, where the function $\sum_j^n c_j x_j$ (the total selling price) is maximal.

The basic problem of linear programming naturally arises here.

The Transportation Problem

Suppliers $A_1, \dots, A_m$ carry the amounts $a_1, \dots, a_m$, respectively, of a certain product.

Customers $B_1, \dots, B_n$ need the amounts $b_1, \dots, b_n$, respectively, of the same product.

It is also given that $\sum_i^m a_i = \sum_j^n b_j$. Let x_{ij} be the amount of product that is transported from A_i to B_j and c_{ij}, the cost to deliver a unit of product from A_i to B_j.

The Transportation Problem

The following conditions must hold:

$$\sum_{j=1}^{n} x_{ij} = a_i, \ \sum_{i=1}^{m} x_{ij} = b_j, \ \ x_{ij} \geq 0.$$

They define a convex polyhedron in the (mn)-space with coordinates x_{ij}.

The problem is to minimize the function $\sum_{i,j} c_{ij} x_{ij}$ on this polyhedron.

The Simplex Method

Sliding by the edges of M in the direction of the increase of f, while possible. The movement ends at a vertex of the maximum.