Linear Algebra

Lecture 2: Euclidean Affine Geometry

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Affine Hull

Suppose $M \subset \mathbb{A}$ and $A_0 \in M$. Then a plane

$$P = A_0 + \langle \overline{A_0 X} \mid X \in M \rangle$$

is the smallest plane that contains M.

This plane is called an affine hull of M and is denoted by $\mathrm{aff}(M)$.

Euclidean Affine Space $\mathbb{E}^n$

Euclidean Affine Space $\mathbb{E}^n$ is an affine space over $V = \mathbb{R}^n$ equipped with a standard Euclidean inner product

$$(\cdot,\cdot)\colon V \times V \to \mathbb{R},\ (u,v) = u_1 v_1 + \ldots + u_n v_n$$

and a metric (and a norm) on $\mathbb{E}^n$:

$$\rho(x,y) := \sqrt{(x-y,x-y)} := \|x-y\|.$$

Euclidean Space: Facts

(F1) $\rho(A, B) \geq 0$, and
$\rho(A, B) = 0 \Leftrightarrow A = B$

(F2) Pythagorean Theorem: If $u \perp v$, that is, $(u, v) = 0$, then $w = u - v$ satisfies the formula: $\|w\|^2 = \|u\|^2 + \|v\|^2$.

(F3) Cauchy–Bunyakovsky–Schwarz Inequality: $|(u, v)| \leq \|u\| \cdot \|v\|$

(F4) Triangle Inequality:
$\rho(A, B) + \rho(B, C) \geq \rho(A, C)$

Euclidean Space: Facts

$[A, B] := \{X \mid \overline{AX} = \lambda\overline{AB},\ 0 \leq \lambda \leq 1\}$ is called a segment (or closed interval)

(F5) For any nonzero $u, v \in \mathbb{R}^n$ there exist vectors $\mathrm{proj}_u v$ and $\mathrm{ort}_u v$, such that $\mathrm{ort}_u v \perp u$, $\mathrm{proj}_u v$ is proportional to u, and $v = \mathrm{proj}_u v + \mathrm{ort}_u v$.

(F6) $\rho(A, B) + \rho(B, C) = \rho(A, C)$ iff $B \in [A, C]$

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Geometry of Euclidean Affine Plane $\mathbb{E}^2$

Axioms and many facts of 2-dimensional geometry become very easy exercises.

- $\forall\ A \neq B \in \mathbb{E}^2$ there exists a unique line ℓ, passing through A and B.
- For any line ℓ and any point $A \notin \ell$ there exists a unique $\ell_1 \parallel \ell$, s.t. $A \in \ell_1$.
- $\forall \ell$ and $A \in \mathbb{E}^2$ there exists a unique $\ell_1 \perp \ell$, s.t. $A \in \ell_1$.

Geometry of Euclidean Affine Plane $\mathbb{E}^2$

- $A, B, C \in \mathbb{E}^2$ form a triangle if $\dim\left(\text{aff}(\{A, B, C\})\right) = 2$
- $\Leftrightarrow$ three triangle inequalities!
- The barycentric combination $X = \lambda A + \mu B$ belongs to a line AB and divides the interval (A, B) in the ratio $\overline{AX} : \overline{XB} = \mu : \lambda$, i.e. $\lambda \overline{AX} = \mu \overline{XB}$. If $\lambda, \mu \geq 0$, then $X \in [A, B]$.

Euclidean Space: Segment

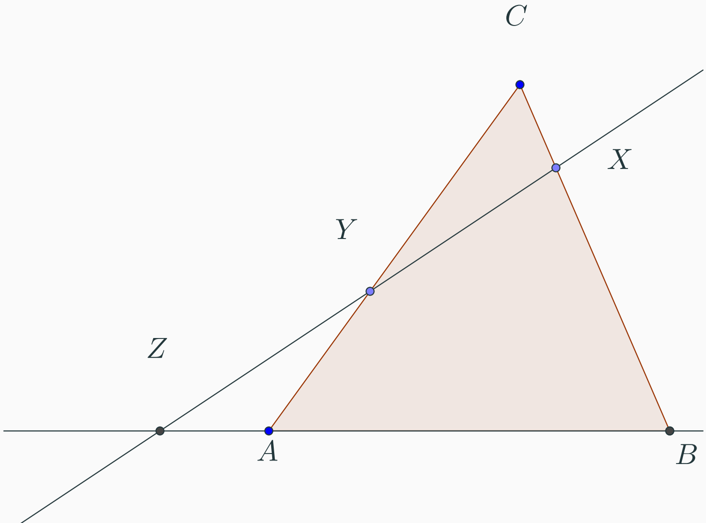
$$
\begin{aligned}
[A, B] &:= \{X \mid \overline{AX} = \lambda \overline{AB},\ 0 \le \lambda \le 1\} = \\
&= \{\alpha A + \beta B \mid \alpha + \beta = 1, \alpha, \beta \ge 0\} = \\
&= \{X \mid \rho(A, X) + \rho(X, B) = \rho(A, B)\}.
\end{aligned}
$$

The Menelaus Theorem

Suppose $A, B, C \in \mathbb{E}^2$ form a triangle, X, Y, Z belong to the intervals BC, CA, AB or their continuations and divide them in the ratio $\lambda : 1$, $\mu : 1$, $\nu : 1$.

Then $\dim\left(\operatorname{aff}\{X, Y, Z\}\right) = 1$ iff $\lambda\mu\nu = -1$.

The Menelaus Theorem: Picture



The Menelaus Theorem: Proof

Proof:

- The matrix of barycentric coordinates of X, Y, Z with respect to A, B, C:

$$\mathrm{Mat}(X, Y, Z) = \begin{pmatrix} 0 & \frac{1}{\lambda+1} & \frac{\lambda}{\lambda+1} \\ \frac{\mu}{\mu+1} & 0 & \frac{1}{\mu+1} \\ \frac{1}{\nu+1} & \frac{\nu}{\nu+1} & 0 \end{pmatrix}$$

- $\dim\left(\mathrm{aff}(\{X, Y, Z\})\right) = 1$ iff $\det \mathrm{Mat}(X, Y, Z) = 0 \Leftrightarrow \lambda\mu\nu = -1.$