Linear Algebra

Lecture 3: Bilinear Forms

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Linear Functions

Suppose V is a vector space with a basis $\{e_1, \ldots, e_n\}$. Linear functions $f\colon V \to \Bbbk$: $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$.

They form a subspace V^* in a space of all $\Bbbk$-valued functions $F(V, \Bbbk)$.

Let $(v_1, \ldots, v_n)$ be the coordinates of v in $\{e_1, \ldots, e_n\}$. Then $f(v) = \sum_{k=1}^{n} v_k f(e_k)$.

Dual Space

V^* is called a dual space.

Let $\{f_1, \dots, f_n\}$ be linear functions, such that $f_i(e_j) = \delta_{ij}$, where $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$.

Then $\{f_1, \dots, f_n\}$ is a basis of V^* and it is called a dual basis.

Examples of Linear Functions

- $f(x) = (a, x) = a_1 x_1 + ... + a_n x_n$.
- $\varphi(f) = f(x_0)$ is a linear function on the set of all $\Bbbk$-valued functions.
- $\varphi(f) = f'(x_0)$ is a linear function on the set of all differentiable functions.
- $\alpha \in C^*[a, b]$, where $\alpha(f) = \int_a^b f(x)dx$
- $\alpha \in \mathrm{Mat}_n^*(\Bbbk)$, where $\alpha(X) = \mathrm{tr}\ X$.

Bilinear Forms

A map $\alpha \colon V \times V \to \Bbbk$ is called a bilinear form, if it is linear in both arguments.

Let $\{e_1, \dots, e_n\}$ be a basis of V, and $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be two vectors. Then

$$\alpha(x, y) = \sum_{i,j=1}^{n} a_{ij} x_i y_j.$$

Matrices of Bilinear Forms

That is, $\alpha(x, y) = \sum_{i,j=1}^{n} a_{ij} x_i y_j = X^T A Y$.

When the basis changes:
$(e_1', \dots, e_n') = (e_1, \dots,\ e_n)C$, coordinates of vectors change too: $CX' = X, CY' = Y$.

Then
$X'^T A' Y' = X^T A Y^T = X'^T (C^T A C) Y$. It implies $A' = C^T A C$.

Examples of Bilinear Forms

- The standard inner product: $(a, b) = a_1 b_1 + ... + a_n b_n$
- $\alpha(f, g) = \int_a^b f(x) g(x) dx$
- $\alpha(X, Y) = \mathrm{tr}\,(XY)$

Kernel and Non-degenerate Forms

The kernel of α:

$\mathrm{Ker}(\alpha) = \{v \in V \mid \alpha(u, v) = 0 \ \forall u \in V\}$.

α is called non-degenerate if $\mathrm{Ker}(\alpha) = 0$.

Clearly,

$\mathrm{Ker}(\alpha) = \{v \mid \alpha(v, e_j) = 0,\ j = 1, \ldots, n\}$.

$\dim \mathrm{Ker}(\alpha) = n - \mathrm{rk}\ A$.

Orthogonal Complement

The orthogonal complement of $U \subset V$ is
$U^\perp = \{v \in V \mid \alpha(u, v) = 0 \ \forall u \in U\}$.

Clearly, $V^\perp = \mathrm{Ker}(\alpha)$.

If α is non-degenerate, then

$\dim U^\perp = \dim V - \dim U \ \mathit{and} \ (U^\perp)^\perp = U$.

Symmetric and Skew-Symmetric Forms

α is called symmetric if $\alpha(x, y) = \alpha(y, x)$, and skew-symmetric if $\alpha(x, y) = -\alpha(y, x)$.

It is equivalent to $A^T = A$ and $A^T = -A$, respectively.

A quadratic form associated to symmetric α is $q(x) = \alpha(x, x)$.