Linear Algebra

Lecture 4: Orthogonalization of Quadratic Forms

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Orthogonal basis

A basis $\{e_1, \dots, e_n\}$ is orthogonal with respect to α if $\alpha(e_i, e_j) = 0$ for all $i \neq j$.

Vectors u and v are orthogonal ($u \perp v$) if $\alpha(u, v) = 0$.

Constructing of orthogonal basis

Let $\{e_1, \ldots, e_n\}$ be a basis in V and A a matrix of α.

Suppose A_k is a matrix of $\alpha\,|_{V_k}$, where $V_k = \langle e_1, \ldots, e_k \rangle$. A number $\delta_k = \det A_k$ is a corner minor of A of order k.

Also, let $V_0 = 0$, $\delta_0 = 1$.

Gram-Schmidt Orthogonalization Procedure: Theorem

If all corner minors are non-zero ($\delta_k \neq 0$, $1 \leq k \leq n$), then $\exists!$ a unique orthogonal basis $\{f_1, \dots, f_n\}$ of V such that

$$f_k \in e_k + V_{k-1}, \quad 1 \leq k \leq n.$$

Also, $q(f_k) = \alpha(f_k, f_k) = \frac{\delta_k}{\delta_{k-1}}$.

Gram-Schmidt Orthogonalization Procedure: Proof

Induction by n. $n = 1$: $q(f_1) = \delta_1 = \frac{\delta_1}{\delta_0}$.

$n > 1$: Let $\{f_1, \dots, f_{n-1}\}$ be the basis for V_{n-1}, that satisfies the conditions.
We construct then

$$f_n = e_n + \sum_{j=1}^{n-1} \lambda_j f_j \in e_n + V_{n-1}.$$

Observe that $q(f_k) = \frac{\delta_k}{\delta_{k-1}},\ k = 1, \dots, n-1$.

Gram-Schmidt Orthogonalization Procedure: Proof

Hence, $\lambda_1, \dots, \lambda_{n-1}$ are determined by the orthogonality condition:

$$0 = \alpha(f_n, f_k) = \alpha(e_n, f_k) + \lambda_k q(f_k).$$

Since $f_n \notin V_{n-1}$, we see that $\{f_1, \dots, f_n\}$ is a basis of V.

Gram-Schmidt Orthogonalization Procedure: Proof

It remains to check that $q(f_n) = \frac{\delta_n}{\delta_{n-1}}$. Consider the transition matrix C: $(f_1, \dots, f_n) = (e_1, \dots, e_n)C$. Moreover, $\det C = 1$ and

$$\det A' = \det (C^T A C) = \det A.$$

Besides, $A' = \mathrm{diag}(q(f_1), \dots, q(f_n))$. It implies $\delta_n = q(f_1) \cdot \ldots \cdot q(f_n)$ and the same for δ_{n-1}. ∎

Normal Form

Let $\Bbbk = \mathbb{C}$. Then, by scaling basis vectors and after a suitable permutation a form $q(x)$ assumes a normal form $x_1^2 + \ldots + x_r^2$, where $r = \mathrm{rk}\, q$ is invariant.

Let $\Bbbk = \mathbb{R}$. Here we obtain $q(x) = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_{k+l}^2$, where $k + l = \mathrm{rk}\, q$ is invariant.

Positive and Negative Definite Quadratic Forms

A quadratic form q is positive definite if $q(x) > 0$ for all $x \neq 0$, and negative definite if $q(x) < 0$ for all $x \neq 0$.

If $q(x) = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_{k+l}^2$, then

$$k = \max_{q|_U > 0} \dim U.$$

Proof: $q \mid_{\langle e_1, \ldots, e_k \rangle} > 0$ and $q \mid_{\langle e_{k+1}, \ldots, e_n \rangle} \leq 0$.

The Law of Inertia

Numbers k and l in a normal form $q(x) = x_1^2 + ... + x_k^2 - x_{k+1}^2 - ... - x_{k+l}^2$ do not depend on a basis (these positive and negative indices of inertia are invariants of $q(x)$).

Jacobi Method for $\mathbb{k} = \mathbb{R}$

If all $\delta_k \neq 0$ for a real $q(x)$, then $l =$ the number of changes of sign in $1, \delta_1, \ldots, \delta_n$.