Linear Algebra

Lecture 2: Affine Spaces

Nikolay V. Bogachev

Moscow Institute of Physics and Technology,
Department of Discrete Mathematics,
Laboratory of Advanced Combinatorics and Network Applicationss

Affine Spaces

A set $\mathbb{A}$ is an affine space over a vector space V (or associated to V) if

- any points $A, B \in \mathbb{A}$ correspond to $\overline{AB} \in V$, s.t. a vectorization map

  $$v_A : \mathbb{A} \to V, \quad X \mapsto \overline{AX}$$

  is bijective,
- and for any points $A, B, C \in \mathbb{A}$ we have $\overline{AB} + \overline{BC} = \overline{AC}$.

Affine Spaces

In other words, each $v \in V$ corresponds to the translation map

$$\tau_v\colon \mathbb{A} \to \mathbb{A}, \quad X \mapsto X + v,$$

such that

- for any points $A, B \in \mathbb{A}$ there exists a unique vector v, s.t. $A + v = B$,
- $\tau_u \circ \tau_v = \tau_{u+v}$ for any $u, v \in V$.

Exercises/Examples

- Any vector space V can be equipped by an affine structure over itself.
- Suppose $A, B, C, D \in \mathbb{A}$ and $\overline{AB} = \overline{CD}$. Then $\overline{AC} = \overline{BD}$.
- A set of all reduced quadratic trinomials $\{x^2 + px + q \mid p, q \in \mathbb{R}\}$ is an affine space over $\mathbb{R}[x]_1$.

Affine Planes or Subspaces

A plane (subspace) in an affine space $\mathbb{A}$ is a set of a form $P := A + U$, where $A \in \mathbb{A}$, and $U \subset V$ is a subspace.

- $\dim P := \dim U$
- a line: $\dim P = 1$
- a hyperplane: $\dim P = n - 1$.

Theorem on $k+1$ points

Given any $k+1$ points in $\mathbb{A}$, there is a plane of $\dim \leq k$, passing through these points.
Moreover, if these points are not contained in any plane of $\dim < k$, then there is a unique $k-\dim$ plane, passing through these points.

Theorem on $k+1$ points

Proof:

- Let $A_0, A_1, \dots, A_k \in \mathbb{A}$. Then $P := A_0 + \langle \overline{A_0A_1}, \overline{A_0A_2}, \dots, \overline{A_0A_k} \rangle$
- If $\dim P = k$, then the vectors $\overline{A_0A_1}, \overline{A_0A_2}, \dots, \overline{A_0A_k}$ are linearly independent and P is unique. ■

$A_0, A_1, \dots, A_k \in \mathbb{A}$ are affinely dependent if they lie in a plane of $\dim < k$, and affinely independent otherwise.

Affine Coordinates

- We can choose a point $O \in \mathbb{A}$ (the origin). Then any point $A \in \mathbb{A}$ is given by its position vector $\overline{OA}$.
- A point O with a basis $\{e_1, \dots, e_n\}$ of V is a frame of an affine space $\mathbb{A}$.
- The coordinates of a point X in the frame $(O; e_1, \dots, e_n)$ equal $(x_1, \dots, x_n)$, where $\overline{OX} = x_1 e_1 + \dots + x_n e_n$.

Affine Coordinates

- This coordinate system in the frame $(O; e_1, \dots, e_n)$ is so called affine coordinate system.
- Coordinates of $A + v$ are equal to the sums of coordinates of A and coordinates of v.
- $\overline{AB} = B - A$.

Solutions of Systems of Linear Equations

Affine planes are sets of solutions of systems of linear equations.

- $\sum_{j=1}^{n} a_{ij}x_j = b_i$, $i = 1, \dots, m.$ (1)
- $(x_1, \dots, x_n)$ are affine coordinates in a frame $(O; e_1, \dots, e_n)$.
- Let A_0 be a solution of the system (1). Then X is a solution iff $\overline{A_0X}$ satisfies the system of homogeneous equations $\sum_{j=1}^{n} a_{ij}x_j = 0$, $i = 1, \dots, m.$

Solutions of Systems of Linear Equations

- Thus, if the system is compatible, then its set of solutions is the plane $A_0 + U$.
- Suppose now $P = A + U$ is a plane.
- U is a set of solutions of a system of homogeneous linear equations.
- Then $A + U$ is a set of solutions of the system with values b_i, that left-hand side assumes at the point A.

Theorem on Relative Position of Planes

Suppose $P_1 = A_1 + U_1$ and $P_2 = A_2 + U_2$.
Then $P_1 \cap P_2 \neq \emptyset$ iff $\overline{A_1 A_2} \in U_1 + U_2$.

Proof:

- $P_1 \cap P_2 \neq \emptyset$ iff $\exists\, u_1 \in U_1$ and $u_2 \in U_2$, s.t. $A_1 + u_1 = A_2 + u_2$.
- That is, $\overline{A_1 A_2} = u_1 - u_2$.
- It is possible iff $\overline{A_1 A_2} \in U_1 + U_2$. ∎

Relative Position of Planes

Suppose $P_1 = A_1 + U_1$ and $P_2 = A_2 + U_2$.

- As it was proved, $P_1 \cap P_2 \neq \emptyset$ iff $\overline{A_1 A_2} \in U_1 + U_2$.
- P_1 and P_2 are called parallel if $U_1 \subset U_2$ or $U_2 \subset U_1$.
- P_1 and P_2 are skew if $P_1 \cap P_2 = \emptyset$ and $U_1 \cap U_2 = 0$.

Barycentric Coordinates

We can define some special linear combinations of points in $\mathbb{A}$.

- Suppose $A_1, \ldots, A_k \in \mathbb{A}$, and $\lambda_1 + \ldots + \lambda_k = 1$.
- Then a barycentric combination $\sum_{j=1}^{k} \lambda_j A_j$ is a point A, s.t. $\overline{OA} = \sum_{j=1}^{k} \lambda_j \overline{OA_j}$.

Barycentric Coordinates

- This definition does not depend on a point O! It is due to the fact that $\sum_{j=1}^{k} \lambda_j = 1$.
- Indeed, $\overline{O'A} = \overline{O'O} + \overline{OA} = \sum_{j=1}^{k} \lambda_j (\overline{O'O} + \overline{OA_j}) = \sum_{j=1}^{k} \lambda_j (\overline{O'A_j})$.
- $\text{center}(A_1, \ldots, A_k) = \frac{1}{k}(A_1 + \ldots + A_k)$ is a center of mass.

Barycentric Coordinates

- Let $A_0, A_1, \ldots, A_n \in \mathbb{A}$ be affinely independent. It is equivalent to linear independence of $\overline{A_0 A_1}, \ldots, \overline{A_0 A_n}$.
- Then any point $X \in \mathbb{A}$ has a unique representation

$$X = \sum_{k=0}^{n} x_k A_k, \quad \sum_{k=0}^{n} x_k = 1.$$

Barycentric Coordinates

- Indeed, we have that

$$\overline{A_0X} = \sum_{k=1}^{n} x_k \overline{A_0A_k}.$$

- It implies, that $x_1, \ldots, x_n$ are the coordinates of $\overline{A_0X}$ in the basis $\{\overline{A_0A_1}, \ldots, \overline{A_0A_n}\}$.
- It remains to take $x_0 = 1 - \sum_{k=1}^{n} x_k$.

Affine Independence and Barycentric Coordinates: Theorem

Points $X_0, X_1, \dots, X_k \in \mathbb{A}$ are affinely independent

$$\text{if and only if}$$

the rank of a matrix $\mathrm{Mat}(X_0, X_1, \dots, X_k)$ of their barycentric coordinates (with respect to $A_0, A_1, \dots, A_n$) equals $k+1$.

Affine Independence and Barycentric Coordinates: Theorem

Proof:

- Let $x_{j0}, x_{j1}, \dots, x_{jn}$ be coordinates of X_j: $\overline{A_0X_j} = \sum_{s=1}^{n} x_{js}\overline{A_0A_s}$.
- We add to the 1st column of $\mathrm{Mat}(X_0, X_1, \dots, X_k)$ the sum of all other columns. After that we can differ the 1st row from all other rows. The rank is invariant.

Affine Independence and Barycentric Coordinates: Theorem

Proof: Thus, we obtain a matrix

$$\begin{pmatrix} 1 & x_{01} & \dots & x_{0n} \\ 0 & x_{11} - x_{01} & \dots & x_{1n} - x_{0n} \\ \dots & \dots & \dots & \dots \\ 0 & x_{k1} - x_{01} & \dots & x_{kn} - x_{0n} \end{pmatrix}$$

Its submatrix is $\mathrm{Mat}(\overline{X_0X_1}, \dots, \overline{X_0X_k})$ of rank equals k.