Linear Algebra

Lecture 2: Affine Maps

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Change of Basis

Suppose $\{e_1, \ldots, e_n\}$ is a basis of V, and we change a basis to $\{e'_1, \ldots, e'_n\}$, where $e'_j = \sum_{i=1}^{n} c_{ij} e_i$.

In matrix form: $(e'_1, \ldots, e'_n) = (e_1, \ldots, e_n)C$, where $C = (c_{ij})$ is the transition matrix.

Then

$x = x_1 e_1 + \ldots + x_n e_n = x'_1 e'_1 + \ldots + x'_n e'_n$,

that is, $x = (e'_1, \ldots, e'_n)X' = (e_1, \ldots, e_n)X$.

Change of Basis and Maps

Using matrix form for bases, we have $(e'_1, \dots, e'_n)X' = (e_1, \dots, e_n)CX' =$ $= (e_1, \dots, e_n)X$. Thus, $CX' = X$.

Suppose $F\colon V \to W$ is a linear map, $\{e_1, \dots, e_n\}$ is a basis of V. Then $F(v) = F \cdot (v_1, \dots, v_n)^T = FV$ in a matrix form, and we change a basis again $(e'_1, \dots, e'_n) = (e_1, \dots, e_n)C$.

Linear Maps and Coordinates

Then

$$F(v) = FV = F'V',$$

and $V = CV'$. It implies

$$FCV' = F'V',$$

thus $F' = FC$.

Affine Maps

Suppose $\mathbb{A}$ and $\mathbb{A}'$ are affine spaces over V and V' (over $\Bbbk$), respectively. Then

$$f\colon \mathbb{A} \to \mathbb{A}'$$

with the property

$$f(A+v) = f(A) + \varphi(v), \quad A \in \mathbb{A}, v \in V$$

for some linear map $\varphi\colon V \to V'$ is called an affine map.

Affine Maps

It implies that

$$\varphi(\overline{AB}) = \overline{f(A)f(B)}, \quad A, B \in \mathbb{A}.$$

That is, a linear map $\varphi\colon V \to V'$ is uniquely determined by f and is called the differential of f. We will denote it by $\varphi = df$

Coordinates

Suppose O, O' are the origins of $\mathbb{A}, \mathbb{A}'$.
Then, using the vectorization
$v_O(X) = \overline{OX}$: $f(\overline{OX}) = \varphi(\overline{OX}) + \overline{O'O}$.
That is, in coordinates $f(x) = \varphi(x) + b$,
where $x = (x_1, \dots, x_n)$ are coordinates
both of $X \in \mathbb{A}$ and $\overline{OX}$ in V.
Thus, if $f(x) = y = (y_1, \dots, y_n)$,

$$y_i = \sum_{j=1}^{n} a_{ij} x_j + b_i, \ i = 1, \dots, n.$$

Composition of Affine Maps

Suppose $f\colon \mathbb{A} \to \mathbb{A}'$ and $g\colon \mathbb{A}' \to \mathbb{A}''$. Then $gf\colon \mathbb{A} \to \mathbb{A}''$ is also an affine map and $d(gf) = dg \cdot df$.

Proof:

$(gf)(A+v) = g(f(A)+df(v)) = g(f(A))+ dg(df(v)) = (gf)(A) + (dg \cdot df)(v)$.

Bijective Affine Maps

$f\colon \mathbb{A} \to \mathbb{A}'$ is bijective iff $df\colon V \to V'$ is bijective.

Proof: Using the origins O and $O' = f(O)$, we have $f(x) = df(x)$. ∎

An isomorphism of affine spaces is bijective affine map.

$\mathbb{A} \simeq \mathbb{A}'$ iff $\dim \mathbb{A} = \dim \mathbb{A}'$.

Images of Affine Maps

Let $f\colon \mathbb{A} \to \mathbb{A}'$ be an affine map. Then the image of $P = A_0 + U$ is $f(P) = f(A_0) + df(U)$, and

$$f\left(\sum_k \lambda_k A_k\right) = \sum_k \lambda_k f(A_k).$$

Proof: Using the vectorization, $a_k = \overline{OA_k}$, and $f(\sum_k \lambda_k A_k) = df(\sum_k \lambda_k a_k) + b = \sum_k (\lambda_k df(a_k) + b) = \sum_k \lambda_k f(A_k)$. ∎

Affine-Linear Functions

A particular case of affine maps are affine-linear functions $f\colon \mathbb{A} \to \mathbb{k}$, s.t.

$$f(A+v) = f(A) + \alpha(v),$$

where $\alpha\colon V \to \mathbb{k}$ is a linear function on V.

Barycentric coordinates are affine-linear functions!

$f \equiv \text{const} \Leftrightarrow df = 0$.