Linear Operators

LA7⋄1. Find the matrix of a linear operator $X \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} X$ in the space $\mathrm{Mat}_2(\mathbb{R})$ with the standard basis.

LA7⋄2. Prove that an eigenspace $V_\lambda(A)$ of an operator A is an invariant subspace for each operator B such that $AB = BA$.

LA7⋄3. Give an example of an operator A on some Euclidean (or Hermitian) vector space such that it has an invariant subspace U and $A(U^\perp) \not\subset U^\perp$ (*it is a counter-example to the Theorem 5.1 in lecture notes for a general operator*).

LA7⋄4. Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of some matrix A. Find the eigenvalues of an operator
(a) $X \mapsto AXA$ in the space $\mathrm{Mat}_n(\mathbb{R})$,
(b) $X \mapsto AXA^{-1}$ in the space $\mathrm{Mat}_n(\mathbb{R})$.

LA7⋄5. Is it true that a matrix of a symmetric operator should be symmetric in all bases?

LA7⋄6. Suppose $f(t) = f_1(t)f_2(t)$ is a decomposition of a polynomial $f(t)$ into the product of two relatively prime polynomials and suppose that $f(A) = 0$ for some linear operator A (over $\mathbb{R}$ or $\mathbb{C}$). Prove that there exists a basis such that

$$A \simeq \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $f_1(A_1) = f_2(A_2) = 0$.

LA7⋄7. Suppose A, B are some linear operators in the same vector space V. Prove that $f_{AB}(t) \equiv f_{BA}(t)$.