Linear Algebra

Lecture 5: Euclidean Vector Spaces

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Euclidean Vector Space

A real vector space V equipped with a positive definite symmetric bilinear form is called Euclidean.

This bilinear form is called an inner product and is denoted by $(\cdot, \cdot)$.

It allows to calculate lengths and angles:

$$\|x\| = \sqrt{(x, x)}, \ \cos\angle(x, y) = \frac{(x, y)}{\|x\| \cdot \|y\|}.$$

Some Examples

$\mathbb{R}^n$ with the standard Euclidean inner product $(x, y) = x_1 y_1 + \ldots + x_n y_n$.

The space $C[a, b]$ with

$$(f, g) = \int_a^b f(x) g(x) dx.$$

Cauchy-Bunyakowski-Schwarz Inequality:

$$|(x, y)| \leq \|x\| \cdot \|y\|.$$

Gram Matrix

The Gram matrix of a system of vectors:

$$G(v_1, \dots, v_k) = \begin{pmatrix} (v_1, v_1) & (v_1, v_2) & \dots & (v_1, v_k) \\ (v_2, v_1) & (v_2, v_2) & \dots & (v_2, v_k) \\ \vdots & \vdots & \ddots & \vdots \\ (v_k, v_1) & (v_k, v_2) & \dots & (v_k, v_k) \end{pmatrix}$$

If $\{e_1, \dots, e_n\}$ is a basis of V, then

$$G(e_1, \dots, e_n) = \mathrm{Mat}((\cdot, \cdot)).$$

Gram Matrix: Theorem

For any $v_1, \ldots, v_k$ in a Euclidean space V
$\det G(v_1, \ldots, v_k) \geq 0$ and $= 0$ iff $v_1, \ldots, v_k$ are linearly dependent.

Proof: $\sum_{j=1}^n \lambda_j a_j = 0 \Rightarrow$ for all $k = 1, \ldots, n$
$\sum_{j=1}^n \lambda_j (a_j, a_k) = 0 \Rightarrow \det G(v_1, \ldots, v_k) = 0$.
If $v_1, \ldots, v_k$ are linearly independent, then they form a basis of a subspace U in V.
Then $\mathrm{Mat}((\cdot, \cdot) \mid_U) = G(v_1, \ldots, v_k) > 0$.

Orthonormal Basis

A basis $\{e_1, \ldots, e_n\}$ is called orthonormal if $(x, y) = x_1 y_1 + \ldots + x_n y_n$.
It is equivalent to any of the following conditions:

- $(x, x) = x_1^2 + \ldots + x_n^2$
- $G(e_1, \ldots, e_n) = E (= \mathrm{Mat}(\mathrm{Id}))$
- $(e_i, e_j) = \delta_{ij}$
- $(e_i, e_j) = 0$ and $\|e_k\| = 1$.

Transition Between Orthonormal Bases

Suppose $\{e_1, \ldots, e_n\}$ is orthonormal and

$$(e'_1, \ldots, e'_n) = (e_1, \ldots, e_n)C.$$

Then the matrix of $(\cdot, \cdot)$ in $\{e'_1, \ldots, e'_n\}$ is

$$C^T E C = C^T C.$$

Therefore, $\{e'_1, \ldots, e'_n\}$ is orthonormal, iff

$$C^T C = E.$$

Orthogonal Matrices

The matrices C, s.t. $C^TC = E$, are called orthogonal. It implies that $\det C = \pm 1$. The definition is equivalent to any of the following conditions:

- $C^{-1} = C^T$
- $CC^T = E$
- $(Cx, Cy) = (x, y)$ for any $x, y \in V$
- rows (and columns) of C are pairwise orthogonal and have length 1.

Orthogonal Projection and Component

Suppose $U \subset V$, then $(\cdot,\cdot)\mid_U > 0$.
It implies that $V = U \oplus U^{\perp}$, which follows that any $x \in V$ can be uniquely written as

$$x = \mathrm{pr}_U x + \mathrm{ort}_U x, \quad \mathrm{pr}_U x \in U, \mathrm{ort}_U x \in U^{\perp}.$$

$\mathrm{pr}_U x$ is the orthogonal projection of x on(to) U, $\mathrm{ort}_U x$ is the orthogonal component of x with respect to U.

Orthogonal Projection and Component

Suppose $\{e_1, \dots, e_k\}$ is an orthogonal basis of $U \subset V$. Then

$$\mathrm{pr}_U x = \sum_{j=1}^{k} \frac{(x, e_j)}{(e_j, e_j)} e_j.$$

Clearly,

$$\mathrm{ort}_U x = x - \mathrm{pr}_U x = x - \sum_{j=1}^{k} \frac{(x, e_j)}{(e_j, e_j)} e_j.$$