Affine and Vector Spaces

LA1⋄1. Suppose ℓ_1 and ℓ_2 are skew lines in the space $\mathbb{R}^3$. Is it true that lines PQ, where $P \in \ell_1$, $Q \in \ell_2$, sweep the whole space?

LA1⋄2. Find a basis of the vector space $V = \{p(x) \in \mathbb{R}_4[x] \mid p'(5) = 0\}$.

LA1⋄3. Find a dimension and a basis of the vector space
(a) of all symmetric matrices $A \in \mathrm{Mat}_n(\mathbb{R})$.
(b) of all skew-symmetric matrices $A \in \mathrm{Mat}_n(\mathbb{R})$.
(c) $\mathfrak{sl}_n(\mathbb{R}) = \{A \in \mathrm{Mat}_n(\mathbb{R}) \mid \mathrm{tr}\, A = 0\}$.

LA1⋄4. Give an example of a finite dimensional space V and three its pairwise transversal subspaces U, W, T (that is, intersecting only at the origin) such that $\dim U + \dim W + \dim T = \dim V$, but $U + W + T \neq V$.

LA1⋄5. Prove that

$$\dim(U + V) = \dim U + \dim V - \dim(U \cap V).$$

Is it true that

$$\dim(U+V+W) = \dim U+\dim V+\dim W-\dim(U\cap V)-\dim(U\cap W)-\dim(W\cap V)+\dim(U\cap V\cap W)?$$

LA1⋄6. Suppose $\dim(U + V) = \dim(U \cap V) + 1$ for some two vector subspaces $U, V \subset \mathbb{R}^n$. Is it true that $U + V$ is equal to one of the subspaces U, V and $U \cap V$ is equal to another?

LA1⋄7. Is it possible that the intersection of the positive orthant

$$\{(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \geq 0\} \subset \mathbb{R}^4$$

with some 2-dimensional plane is a square?

LA1⋄8. Let V be a vector space of dimension n over the finite field $\mathbb{F}_q$ of q elements. How many
(a) vectors (b) bases (c) k-dimensional subspaces
are there in V?

LA1⋄9. Let V be an affine space of dimension n over the finite field $\mathbb{F}_q$ of q elements. How many k-dimensional affine subspaces are there in V?

LA1⋄10. For which $c \in \mathbb{R}$ the hyperplane $\sum x_j = c$ intersects with
(a) the three dimensional cube $I^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x_j| \leq 1\ \forall j\}$?
(b) the four dimensional cube $I^4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid |x_j| \leq 1\ \forall j\}$?
Draw all the polygons and polyhedra respectively that are cut from the cube by such hyperplanes.

LA1⋄11. Prove that the vector space of all continuous functions on $\mathbb{R}$ is infinite dimensional.