Linear Operators

LA5⋄1. Find the matrix of linear operator

(a) of a 2-dimensional rotation on some angle α,

(b) of a 3-dimensional rotation on $2\pi/3$ around the line, which is given by equations $x_1 = x_2 = x_3$ in the standard basis,

(c) $X \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} X$ in the space $\mathrm{Mat}_2(\mathbb{R})$ with the standard basis.

LA5⋄2. Suppose $f(t) = f_1(t)f_2(t)$ is a decomposition of a polynomial $f(t)$ into the product of two relatively prime polynomials and suppose that $f(A) = 0$ for some linear operator A. Prove that there exist such basis that

$$A \simeq \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $f_1(A_1) = f_2(A_2) = 0$.

LA5⋄3. Prove that an eigenspace $V_\lambda(A)$ of an operator A is an invariant subspace for each operator B such that $AB = BA$.

LA5⋄4. Suppose A, B are some linear operators in the same vector space V. Prove that $f_{AB}(t) \equiv f_{BA}(t)$.

LA5⋄5. Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of some matrix A. Find the eigenvalues of an operator

(a) $X \mapsto AX^tA$ in the space $\mathrm{Mat}_n(\mathbb{R})$,

(b) $X \mapsto AXA^{-1}$ in the space $\mathrm{Mat}_n(\mathbb{R})$.

LA5⋄6. Find the Jordan Form of a matrix A and give a geometric description of the corresponding linear operator, if

(a) $A^2 = E$,

(b) $A^2 = A$.

LA5⋄7. Find the Jordan Form of a matrix A, if $A^3 = A^2$.