Linear Algebra

Lecture 5: Volumes and Distances in Euclidean Spaces

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Euclidean Affine Space

An affine space $\mathbb{A}$ over a Euclidean vector space V is also called Euclidean.

Then we can define
$\rho(x, y) = \sqrt{(x - y, x - y)}$.

Suppose $M, N \subset \mathbb{A}$. Then

$$\rho(M, N) = \inf_{m \in M, n \in N} \rho(m, n).$$

Orthogonal Projection and Component

Suppose $U \subset V$, then $(\cdot, \cdot)\,|_U > 0$ and $V = U \oplus U^{\perp}$, where for any $x \in V$

$$x = \mathrm{pr}_U x + \mathrm{ort}_U x, \quad \mathrm{pr}_U x \in U, \mathrm{ort}_U x \in U^{\perp}.$$

Suppose $\{e_1, \dots, e_k\}$ is an orthogonal basis of $U \subset V$. Then

$$\mathrm{pr}_U x = \sum_{j=1}^{k} \frac{(x, e_j)}{(e_j, e_j)} e_j, \quad \mathrm{ort}_U x = x - \mathrm{pr}_U x.$$

Distance Between Points and Subspaces

Let $x \in \mathbb{A}$ be a point and $U \subset V$ be a vector subspace. Then $\rho(x, U) = \|\mathrm{ort}_U x\|$.

Proof: Suppose $x = y + v$, where $y \neq \mathrm{pr}_U x \in U$, is an arbitrary point (vector) and $v = \mathrm{ort}_U x - u$, $u = -\mathrm{pr}_U v \in U$. Then

$$\rho(x, y) = \|v\| = \sqrt{\|\mathrm{ort}_U x\|^2 + \|u\|^2} > \|\mathrm{ort}_U x\|.$$

Distance Between Points and Subspaces

Let $x \in \mathbb{A}$ and $U = \langle e_1, \dots, e_k \rangle \subset V$. Then

$$(\rho(x, U))^2 = \frac{\det G(e_1, \dots, e_k, x)}{\det G(e_1, \dots, e_k)}.$$

Proof: $x \in U \Rightarrow \rho(x, U) = 0$ and $\det G(e_1, \dots, e_k, x) = 0$. Else, $\mathrm{ort}_U x \neq 0$ and (orthogonalization for $U \oplus \langle x \rangle$)

$$\|\mathrm{ort}_U x\|^2 = (\mathrm{ort}_U x, \mathrm{ort}_U x) = \frac{\delta_{k+1}}{\delta_k} =$$

$$= \frac{\det G(e_1, \dots, e_k, x)}{\det G(e_1, \dots, e_k)}$$

Volumes of Parallelepipeds

An n-dimensional parallelepiped on vectors $v_1, \dots, v_n$ in a Euclidean space is $P(v_1, \dots, v_n) = \{\sum_{j=1}^{n} x_j v_j \mid 0 \le x_j \le 1\}$.

Its base is an $(n-1)$-dim $P(v_1, \dots, v_{n-1})$ and its height is $\|\mathrm{ort}_{\langle v_1, \dots, v_{n-1}\rangle} v_n\|$.

The volume of $P(v_1, \dots, v_n)$ is

$$\mathrm{Vol}\, P(v_1, \dots, v_n) = \mathrm{Vol}\, P(v_1, \dots, v_{n-1}) \cdot \|\mathrm{ort}_{\langle v_1, \dots, v_{n-1}\rangle} v_n\|$$

and $\mathrm{Vol}\, P(v) = \|v\|$.

Volume Formulas

$$(\mathrm{Vol}\ P(v_1, \dots, v_n))^2 = \det G(v_1, \dots, v_n).$$

Proof: By induction ($n = 1$ is trivial):

$$\begin{aligned}
&(\mathrm{Vol}\ P(v_1, \dots, v_n))^2 = \\
&= (\mathrm{Vol}\ P(v_1, \dots, v_{n-1}))^2 \cdot \|\mathrm{ort}_{\langle v_1, \dots, v_{n-1} \rangle} v_n\|^2 = \\
&= \det G(v_1, \dots, v_{n-1}) \cdot \frac{\det G(v_1, \dots, v_{n-1}, v_n)}{\det G(v_1, \dots, v_{n-1})} = \\
&= \det G(v_1, \dots, v_n).
\end{aligned}$$

Volume Formulas

Suppose $v_1, \dots, v_n$ are expressed via the orthonormal basis by the matrix A: $(v_1, \dots, v_n) = (e_1, \dots, e_n)A$. Then $\mathrm{Vol}\, P(v_1, \dots, v_n) = |\det A|$.

Proof: This follows from

$$G(v_1, \dots, v_n) = A^T E A = A^T A,$$

which implies that

$$\det G(v_1, \dots, v_n) = (\det A)^2.$$