Quadratic Forms and Symmetric Bilinear Functions

LA4⋄1. Find the matrix of a bilinear function in new basis (e_1', e_2', e_3'), if it has the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

in the old basis (e_1, e_2, e_3), where

$$e_1' = e_1 - e_2, \quad e_2' = e_1 + e_3, \quad e_3' = e_1 + e_2 + e_3.$$

LA4⋄2. Prove that the determinant of a skew-symmetric integral matrix is a square of some integer number.

LA4⋄3. Suppose f is a skew-symmetric bilinear function on a space V, W is a subspace of V and $W^\perp$ is its orthogonal complement with respect to f. Prove that $\dim W - \dim(W \cap W^\perp)$ is an even number.

LA4⋄4. Are the bilinear functions with matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

isomorphic to each other?

LA4⋄5. Find all $\lambda \in \mathbb{R}$ for which the quadratic form

$$q(x) = 5x_1^2 + x_2^2 + \lambda x_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$$

is positive definite.

LA4⋄6. Find the positive and negative inertial indices of the quadratic form $q(x) = \operatorname{tr} x^2$ on the space $\mathrm{Mat}_n(\mathbb{R})$.