Linear Maps, Linear Functions and Bilinear Forms

LA3⋄1. Suppose that a linear map $A\colon V \to W$ in the bases (v_1, v_2, v_3) of V and (w_1, w_2) of W has the matrix

$$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix}.$$

Find the matrix of A in bases $(v_1, v_1 + v_2, v_1 + v_2 + v_3)$ and $(w_1, w_1 + w_2)$.

LA3⋄2. Suppose f is a nonzero linear function on some vector space V and $U = \ker f$. Prove that $V = U \oplus \langle a \rangle$ for any $a \notin U$.

LA3⋄3. Find the number of all

(a) linear functions $f\colon \mathbb{F}_q^n \to \mathbb{F}_q$.

(b) linear injective maps $f\colon \mathbb{F}_q^n \to \mathbb{F}_q^k$.

(c) linear maps $f\colon \mathbb{F}_q^n \to \mathbb{F}_q^k$.

LA3⋄4. Which of the following functions are bilinear and which are symmetric? Here $A, B \in \mathrm{Mat}_n(\mathbb{R})$.

(a) $f(A, B) = A^T B$

(b) $f(A, B) = \mathrm{tr}\,(AB)$

(c) $f(A, B) = \mathrm{tr}\,(AB - BA)$

(d) $f(A, B) = \mathrm{tr}\,(A + B)$

(e) $f(A, B) = \det(AB)$

(f) $f(A, B) = (AB)_{ij}$

(g) $\alpha(f, g) = \int_a^b f(x)g(x)dx$ on the space $C[a, b]$.

LA3⋄5. Prove that $f(A, B) = \mathrm{tr}\,(AB)$ and $\alpha(f, g) = \int_a^b f(x)g(x)dx$ are non-degenerate forms.

LA3⋄6. Suppose $V = \mathbb{R}[x]_n$. Prove that linear functions $\varphi_0, \varphi_1, \ldots, \varphi_n$ on V, given by $\varphi_k(p) = p^{(k)}(0)$, form a basis of V^*.