Linear Algebra

Lecture 5: Hermitian Spaces

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Sesquilinear Form

A complex vector space V is a space over $\mathbb{C}$.

A sesquilinear form is a function $\alpha\colon V \times V \to \mathbb{C}$, that is linear with respect to the 1st argument and antilinear with respect to the 2nd, that is,

$$\alpha(\lambda_1 x_1 + \lambda_2 x_2, \mu_1 y_1 + \mu_2 y_2) =$$
$$= \overline{\lambda}_1 \mu_1 \alpha(x_1, y_1) + \overline{\lambda}_1 \mu_2 \alpha(x_1, y_2) +$$
$$+ \overline{\lambda}_2 \mu_1 \alpha(x_2, y_1) + \overline{\lambda}_2 \mu_2 \alpha(x_2, y_2).$$

Matrices of Sesquilinear Forms

Let $V = \langle e_1, \dots, e_n \rangle$ and $a_{ij} = \alpha(e_i, e_j)$.
Then $\alpha(x, y) = \sum_{i,j=1}^{n} a_{ij} \overline{x_i} y_j$.

The transition between bases:
$(e'_1, \dots, e'_n) = (e_1, \dots, e_n) C$ and
$A' = C^* A C$, where $C^* = \overline{C}^T, \ A = (a_{ij})$.

α is non-degenerate if
$\mathrm{Ker}(\alpha) := \{ y \mid \alpha(x, y) = 0 \ \forall x \in V \} = 0$.

Hermitian and Quadratic Forms

A sesquilinear form α is called hermitian if $\alpha(x, y) = \overline{\alpha(y, x)} \Leftrightarrow A^* = A$.

A quadratic form $q(x) = \alpha(x, x)$ is positive definite if $q(x) > 0$ for any $x \neq 0$.

Similar theory of orthogonalization methods as for $\Bbbk = \mathbb{R}$!!

Normal form:

$$\alpha(x, y) = \overline{x}_1 y_1 + \ldots + \overline{x}_k y_k - \overline{x}_{k+1} y_{k+1} - \overline{x}_{k+l} y_{k+l},$$

$$q(x) = |x_1|^2 + \ldots + |x_k|^2 - |x_{k+1}|^2 - \ldots - |x_{k+l}|^2.$$

Hermitian Vector Space

A sesquilinear form α is called hermitian if $\alpha(x, y) = \overline{\alpha(y, x)} \Leftrightarrow A^* = A$.

A complex vector space V with a positive definite hermitian form is Hermitian.

This form is also called an inner product and is denoted by $(\cdot, \cdot)$.

Some Examples

$\mathbb{C}^n$ with the standard Hermitian inner product $(x, y) = \overline{x}_1 y_1 + ... + \overline{x}_n y_n$.

The space $C[a, b]$ with

$$(f, g) = \int_a^b \overline{f(x)} g(x) dx.$$

Cauchy-Bunyakowski-Schwarz Inequality:

$$|(x, y)| \leq \|x\| \cdot \|y\|.$$

Some Facts

Inner product allows to calculate lengths and distances:

$$\|x\| = \sqrt{(x,x)},\ \rho(x,y) = \sqrt{(x-y,x-y)}.$$

If $(Cx, Cy) = (x, y)$ then $C^*C = E$. Such C form the group $\mathrm{U}(n, \mathbb{C})$ of unitary matrices (with $|\det C| = 1$).