Linear Algebra

Lecture 2: Affine Transformations

Nikolay V. Bogachev

Moscow Institute of Physics and Technology,
Department of Discrete Mathematics,
Laboratory of Advanced Combinatorics and Network Applicationss

Affine Transformations

Affine map $f\colon \mathbb{A} \to \mathbb{A}$ is called an affine transformation.

Bijective affine transformations form the general affine group $\mathrm{Aff}(\mathbb{A})$.

A differential map $d\colon \mathrm{Aff}(\mathbb{A}) \to \mathrm{GL}(V)$ is a homomorphism. Its kernel is a group $T(\mathbb{A})$ of parallel translations $\tau_v\colon A \mapsto A + v$.

Parallel Translations

For any $f \in \mathrm{Aff}(\mathbb{A})$ and $v \in V$ we have $f\tau_v f^{-1} = \tau_{df(v)}$.

Proof: Suppose $Y = f(X)$, then

$$f\tau_v f^{-1}(Y) = f\tau_v(X) = f(X+v) =$$
$$= f(X) + df(v) = Y + df(v) = \tau_{df(v)}(Y).$$

Using the vectorization map $v_O \colon X \mapsto \overline{OX}$ we have $\mathrm{GL}(V) \subset \mathrm{Aff}(\mathbb{A})$ is a subgroup.

Normal Subgroup

$H \subset G$ is called a normal subgroup, if $gH = Hg$ for any $g \in G$, where $gH = \{gh \mid h \in H\}, Hg = \{hg \mid h \in H\}$. Denoted by $H \lhd G$. Examples:

- Any subgroup of Abelian group
- A subgroup $H \subset G$ with exactly 2 cosets gH is normal
- $T(\mathbb{A}) \lhd \mathrm{Aff}(\mathbb{A})$

Homothety

A homothery with the center $O \in \mathbb{A}$ and coefficient $\lambda \in \Bbbk$ is

$$H_O^\lambda(O+v) = O + \lambda v.$$

In other words, $H_O^\lambda(X) = Y$, such that $\overline{OY} = \lambda \overline{OX}$.

Clearly, $dH_O^\lambda = \lambda \,\mathrm{Id}$.

Simplices

Suppose, $\{A_0, A_1, \dots, A_n\}$ and $\{B_0, B_1, \dots, B_n\}$ are two systems of affinely independent points in an $n-\dim$ space $\mathbb{A}$. Then, $\exists!$ affine transformation, such that $f(A_j) = B_j$, $j = 0, \dots, n$.

Proof: $\exists!$ linear map φ, such that $\varphi(\overline{A_0 A_j}) = \overline{B_0 B_j}$.
Then $f(x) := \varphi(x) + \overline{A_0 B_0}$.

Notions of Affine Geometry

What are the notions of affine geometry?

- Planes map to planes
- Parallel lines map to parallel lines
- Intervals and segments
- Barycentric combinations, midpoints, centers of mass
- Simplices
- But NOT squares, angles, circles!