FROM GEOMETRY TO ARITHMETICITY OF COMPACT HYPERBOLIC COXETER POLYTOPES

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ABSTRACT. In this paper we prove that any compact hyperbolic Coxeter 3-polytope contains an edge such that the distance between its framing facets is small enough. The same holds for every compact Coxeter $(n > 3)$-polytope that has a 3-dimensional face, which is a Coxeter polytope itself. Furthermore, we provide some applications of the above result to classification of stably reflective Lorentzian lattices.

Keywords: Coxeter polytope, hyperbolic reflection group, reflective Lorentzian lattice, arithmetic group.

§ 1. Introduction

One of the main purposes of our paper is to prove that every compact Coxeter polytope P in hyperbolic 3-space $\mathbb{H}^3$ has a 1-dimensional edge E such that the distance between its *framing facets* (i.e. codimension 1 faces containing the vertices of E but not E itself) is small enough. As a simple consequence, the same holds for every compact Coxeter $(n > 3)$-polytope that has a 3-dimensional face, which is a Coxeter polytope itself. In these cases, a part of P bounded by framing facets and facets containing E will be called a *small ridge* of P.

Furthermore, we show that a geometric analysis of such small ridges of Coxeter polytopes can be a very useful tool for classification of arithmetic hyperbolic reflection groups and reflective Lorentzian lattices.

Recall that Coxeter polytopes (i.e. those whose bounding hyperplanes H_i and H_j either do not intersect or form a dihedral angle of π/n_{ij}, where $n_{ij} \in \mathbb{Z}$, $n_{ij} \geq 2$) are fundamental domains for discrete groups generated by reflections in hyperplanes in spaces of constant curvature. Finite volume Coxeter polytopes in $\mathbb{E}^n$ and $\mathbb{S}^n$ were classified by Coxeter himself in 1933 [16]. In 1967, Vinberg [39] developed his theory of hyperbolic reflection groups, and, in particular, proved an arithmeticity criterion for finite covolume hyperbolic reflection groups. It is known (cf. [44, 32, 2]) that there are only finitely many maximal arithmetic hyperbolic reflection groups in all dimensions $n \geq 2$ and they can exist in $\mathbb{H}^n$ only for $n < 30$.

In order to formulate the main results of our paper, we introduce some notation. Let P be a compact *acute-angled* (i.e. those whose dihedral angles are $\leq \pi/2$) polytope in $\mathbb{H}^n$, E an edge of P, $F_1, \ldots, F_{n-1}$ the facets of P containing E, F_n and F_{n+1} the framing facets of E. A part of P bounded by facets $F_1, \ldots, F_{n+1}$ is called a *ridge associated with* E, and the number $\cosh \rho(F_n, F_{n+1})$ is its *width* (here $\rho(\,\cdot\,,\cdot\,)$ is the hyperbolic metric). Every ridge corresponds to a set $\overline{\alpha} = \{\alpha_{ij}\}$[1], where α_{ij} is the angle between the facets F_i and F_j.

Theorem A. *Every compact Coxeter polytope in the hyperbolic 3-space $\mathbb{H}^3$ contains a ridge of width less than $\mathbf{t}_{\overline{\alpha}}$, where $\mathbf{t}_{\overline{\alpha}}$ is the number depending on the set $\overline{\alpha}$ only, and*

$$\max_{\overline{\alpha}}\{\mathbf{t}_{\overline{\alpha}}\} = t_{(\pi/5,\pi/3,\pi/3,\pi/2,\pi/2)} < 5.75.$$

[1] In [21], the ridge associated to the edge E was called the ridge of type $\overline{\alpha}$.

Corollary 1. *Let $P \subset \mathbb{H}^{n\geq 4}$ be a compact Coxeter polytope, and let P' be a 3-dimensional face of P, which is a Coxeter polytope itself* [2]*. Then P has a ridge of width < 5.75.*

In order to classify reflective Lorentzian lattices and to prove finiteness of arithmetic hyperbolic reflection groups, Nikulin proved[3] (cf. [27, Lemma 3.2.1] and the proof of [29, Theorem 4.1.1]) that every finite volume acute-angled polytope in $\mathbb{H}^n$ has a facet F such that $\cosh \rho(F_1, F_2) \leq 7$ for any facets F_1 and F_2 of P adjacent to F. It implies that every compact (even finite volume) acute-angled polytope $P \subset \mathbb{H}^n$ contains a ridge of width ≤ 7 (for the case $n = 3$ cf. [8, Prop. 2.1]).

Remark 1. *The result in Theorem A is essentially new. An explicit formula for $\mathbf{t}_{\overline{\alpha}}$ is available in Theorem 3.1. This new bound $\mathbf{t}_{\overline{\alpha}}$ is much more efficient than Nikulin's estimate*[4].

Remark 2. *In a recent paper by the author, it was shown that every compact arithmetic Coxeter polytope in $\mathbb{H}^3$ with ground field $\mathbb{Q}$ contains a ridge of width < 4.14, cf.* [8, Theorem 1.1]*, however, due to the small technical mistake the correct bound should be 4.98. We shall discuss it in § 6. Notice that Theorem A is much more general, since arithmetic Coxeter polytopes in $\mathbb{H}^3$ with ground field $\mathbb{Q}$ can have dihedral angles $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/6$ only.*

We shall say that a ridge E in a compact acute-angled polytope P is *right-angled* if $\alpha_{ij} = \pi/2$ for every $1 \leq i < j \leq n + 1$.

Theorem B. *Let P be a compact Coxeter polytope in $\mathbb{H}^{n\geq 3}$, O the interior point of P, and let E be the outermost edge from O. If an associated ridge E is right-angled then it has width < 2. In particular, any compact right-angled Coxeter polytope has a ridge of width < 2.*

A Lorentzian lattice L is said to be *reflective* if its automorphism group is up to finite index genereted by reflections, and *stably reflective*, if the same group is up to finite index generated by *stable reflections*. For the detailed discussion and precise definitions of arithmetic hyperbolic reflection groups and reflective Lorentzian lattices see § 5.

In order to formulate the third main result of our paper, we introduce some notation:

1) $[C]$ is a quadratic lattice whose inner product in some basis is given by a symmetric matrix C;
2) $d(L) := \det C$ is the discriminant of the lattice $L = [C]$;
3) $L \oplus M$ is the orthogonal sum of the lattices L and M.

Theorem C. *Every maximal stably reflective Lorentzian lattice L of signature $(3, 1)$ over $\mathbb{Z}[\sqrt{2}]$ is isomorphic to one of the following list:*

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[2] In [9], Bogachev and Kolpakov proved that each face of a quasi-arithmetic Coxeter polytope, which is itself a Coxeter polytope, is also quasi-arithmetic, and also provided a sufficient condition for a codimension 1 face to be arithmetic. A large number of Coxeter polytopes and their faces was studied, using a computer program `PLoF` [10]. It turns out that it is a very usual situation that a Coxeter polytope has many faces, which also are Coxeter polytopes. This means that a condition in Corollary 1 is not so unnatural.

[3] We present this assertion in a form convenient for us, although it was not formulated in this way anywhere. In Nikulin's papers, the lengths squared of the facet normals are equal to (-2) and, therefore, his bound appears in the form $(\delta, \delta') \leq 14$.

[4] The author used it (cf. [8]) in order to classify stably reflective Lorentzian lattices over $\mathbb{Z}$ of signature $(3, 1)$. Using Nukulin's result, one gets around 1000 candidate lattices to be combed through and checked for reflectivity, while using $\mathbf{t}_{\overline{\alpha}}$ leaves us with no more than 50 candidates.

No.	L	*# facets*	$d(L)$
1	$[-1-\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	5	$-1-\sqrt{2}$
2	$[-1-2\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	6	$-1-2\sqrt{2}$
3	$[-5-4\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	5	$-5-4\sqrt{2}$
4	$[-11-8\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	17	$-11-8\sqrt{2}$
5	$[-\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	6	$-\sqrt{2}$
6	$\begin{bmatrix} 2 & -1 & -\sqrt{2} \\ -1 & 2 & \sqrt{2}-1 \\ -\sqrt{2} & \sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix} \oplus [1]$	6	$-\sqrt{2}$
7	$[-7-5\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	5	$-7-5\sqrt{2}$

(Here, "# facets" denotes the number of facets of the fundamental Coxeter polytope for the maximal arithmetic hyperbolic reflection subgroup $\mathcal{O}_r(L)$, preserving L.)

The author hopes that analysing small ridges can become a useful tool for classifying not only stably reflective Lorentzian lattices, but reflective lattices in general.

The paper is organised as follows. In § 2 we provide some preliminary results. Then, § 3 is devoted to the proof of Theorem A (the proof of Corollary 1 is presented in § 3.3) and § 4 is devoted to the proof of Theorem B.

The proof of Theorem A is based on Theorem 2.1 (where an explicit upper bound for the length of the outermost edge of a compact acute-angled polytope in $\mathbb{H}^3$ is obtained) and Theorem 3.1 (with an explicit formula for $\mathbf{t}_{\overline{\alpha}}$). A more detailed plan of the proof of Theorem A is described in § 3.2.

Some definitions and facts concerning arithmetic hyperbolic reflection groups and reflective Lorentzian lattices are collected in § 5. Finally, § 6 is a description of applications of Theorem A to classication of stably reflective Lorentzian lattices and § 7 contains the proof of Theorem C.

Acknowledgments. The author is grateful to Daniel Allcock and Sasha Kolpakov for valuable discussions, helpful remarks and suggestions, and to Stepan Alexandrov for remarks and corrections. The author is also thankful to Institut des Hautes Études Scientifiques — IHES, and especially to Fanny Kassel, for their hospitality while this work was carried out. This work was supported by the grant of RFBR according to the research project 18-31-00427.

§ 2. Preliminaries

2.1. Hyperbolic Lobachevsky space and convex polytopes. Let $\mathbb{E}^{n,1}$ be the $(n+1)$-dimensional pseudo-Euclidean real *Minkowski space* equipped with an inner product

$$(x, y) = -x_0y_0 + x_1y_1 + \ldots + x_ny_n$$

of signature $(n, 1)$. A *vector model* of the n-*dimensional hyperbolic Lobachevsky space* $\mathbb{H}^n$ is the above component of the standard hyperboloid lying in the *future light cone*:

$$\mathbb{H}^n = \{x \in \mathbb{E}^{n,1} \mid (x, x) = -1, x_0 > 0\}.$$

The points of $\mathbb{H}^n$ are called the *proper points*. The *points at infinity* (or on *the boundary* $\partial\mathbb{H}^n$) in this model correspond to *isotropic one-dimensional subspaces* of $\mathbb{E}^{n,1}$, this is, such vectors x that $(x, x) = 0$.

The hyperbolic metric ρ is given by

$$\cosh \rho(x, y) = -(x, y).$$

Let $\mathrm{O}_{n,1}(\mathbb{R})$ be the group of orthogonal transformations of the space $\mathbb{E}^{n,1}$, and let $\mathrm{PO}_{n,1}(\mathbb{R})$ be its subgroup of index 2 preserving $\mathbb{H}^n$. The group $\mathrm{PO}_{n,1}(\mathbb{R}) \simeq \mathrm{Isom}(\mathbb{H}^n)$ is the *isometry group* of the hyperbolic n-space $\mathbb{H}^n$.

Suppose that $e \in \mathbb{E}^{n,1}$ is a unit vector (i.e. $(e, e) = 1$). Then the set

$$H_e = \{x \in \mathbb{H}^n \mid (x, e) = 0\}$$

is a *hyperplane* in $\mathbb{H}^n$, and it divides the entire space into *half-spaces*

$$H_e^- = \{x \in \mathbb{H}^n \mid (x, e) \leq 0\}, \qquad H_e^+ = \{x \in \mathbb{H}^n \mid (x, e) \geq 0\}.$$

An orthogonal transformation which is given by the formula

$$\mathcal{R}_e(x) = x - 2(e, x)e,$$

is called the *reflection in hyperplane* H_e, which is called the *mirror* of $\mathcal{R}_e$.

Definition 2.1. *A convex polytope in $\mathbb{H}^n$ is an intersection of finitely many half-spaces that has non-empty interior. A generalized convex polyhedron is an intersection (with non-empty interior) of a family (possibly infinite) of half-spaces such that any ball intersects only finitely many of their boundary hyperplanes.*

Definition 2.2. *A generalized convex polyhedron is said to be acute-angled if all its dihedral angles do not exceed $\pi/2$. A generalized convex polyhedron is called a Coxeter polyhedron if all its dihedral angles are of the form π/k, where $k \in \{2, 3, 4, \ldots, +\infty\}$.*

It is known that the fundamental domains of discrete reflection groups are generalized Coxeter polyhedra (see [39, 45]).

A convex polytope has finite volume if and only if it is equal to the convex hull of finitely many points of the closure $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$. If a polytope is compact then it is a convex hull of finitely many proper points of $\mathbb{H}^n$.

It is also known that compact acute-angled polytopes and, in particular, compact Coxeter polytopes in $\mathbb{H}^n$ are *simple*, that is, every vertex belongs to exactly n facets (and n edges). This fact will be important for the proof of Corollary 1.

2.2. Bounds for the length of the outermost edge for a compact acute-angled polytope in $\mathbb{H}^3$. In this subsection P denotes a compact acute-angled polytope in the three-dimensional Lobachevsky space $\mathbb{H}^3$. Following Nikulin [29, Theorem 4.1.1], we consider an interior point O in P. Let E be *the outermost*[5] *edge from it* and V_1 and V_2 be the vertices of E.

Let E_1 and E_3 be the edges of the polytope P outgoing from the vertex V_1 and let E_2 and E_4 be the edges outgoing from V_2 such that the edges E_1 and E_2 lie in the face F_1. The length of the edge E is denoted by a, and the plane angles between the edges E_j and E are denoted by α_j (see Figure 1).

Denote by V_1I, V_2I, V_1J, V_2J the bisector of angles α_1, α_2, α_3, α_4, respectively. Let h_I and h_J be the distances from the points I and J to the edge E.

The next theorem was proved by the author (Bogachev, 2019, [8, Theorem 2.1]), but we should note that the statement is slightly corrected.

[5] In an acute-angled polytope, the distance from the interior point to the face (of any dimension) is equal to the distance to the plane of this face.

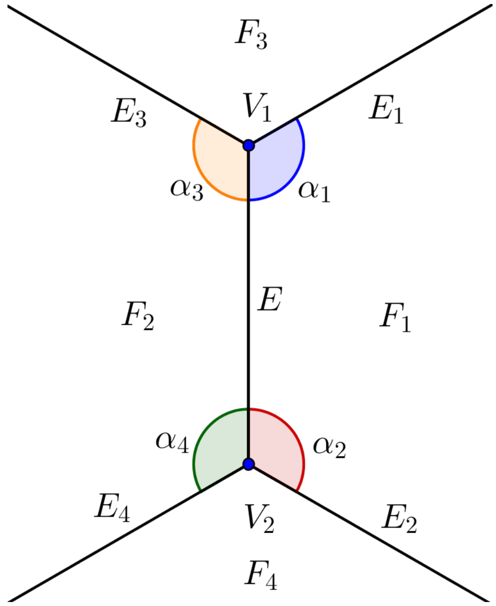

FIGURE 1. The outermost edge

Theorem 2.1. *If $h_J \leqslant h_I$, then the length of the outermost edge satisfies the inequality*

$$a < \operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_3/2)}\right) + \operatorname{arcsinh}\left(\frac{\cos(\alpha_{12}/2)}{\tan(\alpha_4/2)}\right).$$

Proof. See [8, Theorem 2.1] and note that $\tanh(\log(\cot(\alpha_{12}/4))) = \cos(\alpha_{12}/2)$. ∎

Let us introduce the following notation:

$$F_{i,j}(\overline{\alpha}) := \operatorname{arcsinh}\left(\frac{\cos\left(\frac{\alpha_{12}}{2}\right)}{\tan\left(\frac{\alpha_i}{2}\right)}\right) + \operatorname{arcsinh}\left(\frac{\cos\left(\frac{\alpha_{12}}{2}\right)}{\tan\left(\frac{\alpha_j}{2}\right)}\right).$$

Corollary 2. *The following inequality holds:*

$$\cosh a < \max\{\cosh F_{1,2}(\overline{\alpha}), \cosh F_{3,4}(\overline{\alpha})\}.$$

2.3. Auxiliary lemmas.

Lemma 2.1. *The following relations are true:*
(i) $\alpha_{12} + \alpha_{23} + \alpha_{13} > \pi, \quad \alpha_{12} + \alpha_{24} + \alpha_{14} > \pi$;
(ii)

$$\cos\alpha_1 = \frac{\cos\alpha_{23} + \cos\alpha_{12}\cdot\cos\alpha_{13}}{\sin\alpha_{12}\cdot\sin\alpha_{13}}, \qquad \cos\alpha_2 = \frac{\cos\alpha_{24} + \cos\alpha_{12}\cdot\cos\alpha_{14}}{\sin\alpha_{12}\cdot\sin\alpha_{14}},$$
$$\cos\alpha_3 = \frac{\cos\alpha_{13} + \cos\alpha_{12}\cdot\cos\alpha_{23}}{\sin\alpha_{12}\cdot\sin\alpha_{23}}, \qquad \cos\alpha_4 = \frac{\cos\alpha_{14} + \cos\alpha_{12}\cdot\cos\alpha_{24}}{\sin\alpha_{12}\cdot\sin\alpha_{24}}.$$

Proof. See [8, Lemma 2.1]. ∎

Lemma 2.2. *The following expression for $\cosh F_{i,j}(\overline{\alpha})$ holds:*

$$\frac{2\cos^2\left(\frac{\alpha_{12}}{2}\right)\cos\left(\frac{\alpha_i}{2}\right)\cos\left(\frac{\alpha_j}{2}\right) + 2\sqrt{\cos^2\left(\frac{\alpha_{12}}{2}\right)\cos^2\left(\frac{\alpha_i}{2}\right) + \sin^2\left(\frac{\alpha_i}{2}\right)}\sqrt{\cos^2\left(\frac{\alpha_{12}}{2}\right)\cos^2\left(\frac{\alpha_j}{2}\right) + \sin^2\left(\frac{\alpha_j}{2}\right)}}{\sqrt{1-\cos\alpha_i}\sqrt{1-\cos\alpha_j}}.$$

Proof. Using the formula

$$\cosh(\operatorname{arcsinh} x + \operatorname{arcsinh} y) = xy + \sqrt{1+x^2}\sqrt{1+y^2},$$

we obtain that

$$\cosh F_{i,j}(\overline{\alpha}) = \cosh\left(\operatorname{arcsinh}\left(\frac{\cos\left(\frac{\alpha_{12}}{2}\right)}{\tan\left(\frac{\alpha_i}{2}\right)}\right) + \operatorname{arcsinh}\left(\frac{\cos\left(\frac{\alpha_{12}}{2}\right)}{\tan\left(\frac{\alpha_j}{2}\right)}\right)\right) =$$

$$= \frac{\cos^2\left(\frac{\alpha_{12}}{2}\right)\cos\left(\frac{\alpha_i}{2}\right)\cos\left(\frac{\alpha_j}{2}\right) + \sqrt{\cos^2\left(\frac{\alpha_{12}}{2}\right)\cos^2\left(\frac{\alpha_i}{2}\right) + \sin^2\left(\frac{\alpha_i}{2}\right)}\sqrt{\cos^2\left(\frac{\alpha_{12}}{2}\right)\cos\left(\frac{\alpha_j}{2}\right) + \sin^2\left(\frac{\alpha_j}{2}\right)}}{\sin\left(\frac{\alpha_i}{2}\right)\sin\left(\frac{\alpha_j}{2}\right)} =$$

$$= \frac{2\cos^2\left(\frac{\alpha_{12}}{2}\right)\cos\left(\frac{\alpha_i}{2}\right)\cos\left(\frac{\alpha_j}{2}\right) + 2\sqrt{\cos^2\left(\frac{\alpha_{12}}{2}\right)\cos^2\left(\frac{\alpha_i}{2}\right) + \sin^2\left(\frac{\alpha_i}{2}\right)}\sqrt{\cos^2\left(\frac{\alpha_{12}}{2}\right)\cos^2\left(\frac{\alpha_j}{2}\right) + \sin^2\left(\frac{\alpha_j}{2}\right)}}{\sqrt{1-\cos\alpha_i}\sqrt{1-\cos\alpha_j}}.$$

While transforming the expressions above, we use the half-angle formulae, where appropriate. ■

§ 3. Proof of Theorem A

3.1. Explicit formula for $\mathbf{t}_{\overline{\alpha}}$. Let E be the outermost edge of a compact Coxeter polytope P. Consider the set of unit outer normals (u_1, u_2, u_3, u_4) to the facets F_1, F_2, F_3, F_4. Note that this vector system is linearly independent. Its Gram matrix is

$$G(u_1,u_2,u_3,u_4) = \begin{pmatrix} 1 & -\cos\alpha_{12} & -\cos\alpha_{13} & -\cos\alpha_{14} \\ -\cos\alpha_{12} & 1 & -\cos\alpha_{23} & -\cos\alpha_{24} \\ -\cos\alpha_{13} & -\cos\alpha_{23} & 1 & -T \\ -\cos\alpha_{14} & -\cos\alpha_{24} & -T & 1 \end{pmatrix},$$

where $T = |(u_3,u_4)| = \cosh\rho(F_3,F_4)$ is width of E in the case where the facets F_3 and F_4 diverge. Recall that otherwise $T \leq 1$, and we do not need to consider this case separately. Let us denote by G_{ij} the algebraic complements of the elements of the matrix $G = G(u_1,u_2,u_3,u_4)$.

We denote by $F(\overline{\alpha})$ the corresponding $F_{i,j}(\overline{\alpha})$, depending on $h_J \leq h_I$ or $h_I \leq h_J$ (see Theorem 2.1).

Theorem 3.1. *A small ridge associated with the edge E of a compact Coxeter polytope $P \subset \mathbb{H}^3$ has width T less than*

$$\mathbf{t}_{\overline{\alpha}} = \frac{\cosh F(\overline{\alpha}) \cdot \sqrt{G_{33}G_{44}} - g(\overline{\alpha})}{\sin^2\alpha_{12}},$$

where

$$g(\overline{\alpha}) := \cos\alpha_{12}\cos\alpha_{13}\cos\alpha_{24} + \cos\alpha_{12}\cos\alpha_{14}\cos\alpha_{23} + \cos\alpha_{13}\cos\alpha_{14} + \cos\alpha_{23}\cos\alpha_{24}.$$

Proof. Let $(u_1^*, u_2^*, u_3^*, u_4^*)$ be the basis dual to the basis (u_1,u_2,u_3,u_4). Then u_3^* and u_4^* determine the vertices V_2 and V_1 in the Lobachevsky space. Indeed, the vector v_1 corresponding to the point $V_1 \in \mathbb{H}^3$ is uniquely determined (up to scaling) by the conditions $(v_1,u_1) = (v_1,u_2) = (v_1,u_3) = 0$. Note that the vector u_4^* satisfies the same conditions. Therefore, the vectors v_1 and u_4^* are proportional. Hence,

$$\cosh a = \cosh\rho(V_1,V_2) = -(v_1,v_2) = -\frac{(u_3^*,u_4^*)}{\sqrt{(u_3^*,u_3^*)(u_4^*,u_4^*)}}.$$

It is known that $G(u_1^*, u_2^*, u_3^*, u_4^*) = G(u_1, u_2, u_3, u_4)^{-1}$, whence it follows that $\cosh a$ can be expressed in terms of the algebraic complements G_{ij} (recall that G_{ij} is computed with a sign $(-1)^{i+j}$) of the elements of the matrix $G = G(u_1, u_2, u_3, u_4)$:

$$\cosh a = -\frac{(u_3^*, u_4^*)}{\sqrt{(u_3^*, u_3^*)(u_4^*, u_4^*)}} = \frac{G_{34}}{\sqrt{G_{33}G_{44}}}.$$

Theorem 2.1 implies that

$$\cosh a < \cosh F(\overline{\alpha}).$$

It follows that

$$\frac{G_{34}}{\sqrt{G_{33}G_{44}}} < \cosh F(\overline{\alpha}). \tag{1}$$

For every $\overline{\alpha}$, we obtain in this way a linear inequality with respect to the number T. Indeed,

$$G_{34} = T(1 - \cos^2 \alpha_{12}) + g(\overline{\alpha}) = T \cdot \sin^2 \alpha_{12} + g(\overline{\alpha}) < \cosh F(\overline{\alpha}) \cdot \sqrt{G_{33}G_{44}},$$

which finishes the proof. ∎

3.2. Proof of Theorem A. In order to prove Theorem A it remains to show that

$$\max_{\overline{\alpha}} \mathbf{t}_{\overline{\alpha}} = \mathbf{t}_{(\pi/5,\pi/3,\pi/3,\pi/2,\pi/2)} < 5.75.$$

Taking into account Lemma 2.1, (i), we can see that only one or two angles α_{ij} can be equal to π/k, where $k \geq 6$. Moreover, any triple of angles around one of the vertices of the edge E contains $\pi/2$.

The plan of the proof. Without loss of generality, our plan consists of separate considering of the following cases:

(1) $\alpha_{12} = \frac{\pi}{k}$, where $k \geq 6$. Due to Proposition 3.1, $\mathbf{t}_{\overline{\alpha}} < 2\sqrt{2}$.
(2) $\alpha_{13} = \frac{\pi}{k}$, where $k \geq 6$. It implies by Lemma 2.1, (i) that $\alpha_{12} = \pi/2$. By Proposition 3.2, we have $\mathbf{t}_{\overline{\alpha}} < 5$.
(3) no α_{ij} is equal to π/k for $k \geq 6$, i.e. $\alpha_{ij} = \pi/2, \pi/3, \pi/4, \pi/5$. This gives us 67 different possibilities for a small ridge, and 43 of them are combined by the fact that $\alpha_{12} = \pi/2$. In this case, we use Proposition 3.2 again: $\mathbf{t}_{\overline{\alpha}} < 5$.
(4) It remains to calculate $\mathbf{t}_{\overline{\alpha}}$ for 24 different types of a small ridge. It was done using the program `SmaRBA` (**Sma**ll **R**idges, **B**ounds and **A**pplications, see [11]) written in `Sage` computer algebra system. The result is presented as the list of Coxeter diagrams in Table 1.

In order to obtain upper bounds for $\mathbf{t}_{\overline{\alpha}}$, we shall use Theorem 3.1 and (see Corollary 2)

$$a < F(\overline{\alpha}) = \max\{F_{1,2}(\overline{\alpha}), F_{3,4}(\overline{\alpha})\}.$$

Remark 3. *Note that if one computes bounds for a large (but finite) number of ridges then it can be much more efficient to verify in each case whether $h_J \leq h_I$ or $h_I \leq h_J$.*

Proposition 3.1. *Suppose that $\alpha_{12} = \frac{\pi}{k}$, where $k \geq 6$. Then $\mathbf{t}_{\overline{\alpha}} < 2\sqrt{2}$.*

Proof. Due to Lemma 2.1, (i), we can see that all other $\alpha_{ij} = \pi/2$ and $\overline{\alpha} = \left(\frac{\pi}{k}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$. In this case we have that $F(\overline{\alpha}) = F_{1,2}(\overline{\alpha}) = F_{3,4}(\overline{\alpha})$ and

$$g\left(\frac{\pi}{k}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = 0, \qquad \sqrt{G_{33}G_{44}} = \sin^2\left(\frac{\pi}{k}\right).$$

We have

$$\cosh F\left(\frac{\pi}{k}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = \cosh\left(2\operatorname{arcsinh}\left(\frac{\cos(\pi/2k)}{\tan(\pi/4)}\right)\right) = 2\cos\left(\frac{\pi}{2k}\right)\sqrt{1+\cos^2\left(\frac{\pi}{2k}\right)}.$$

It implies that

$$\mathbf{t}_{\overline{\alpha}} = \frac{2\cos\left(\frac{\pi}{2k}\right)\sqrt{1+\cos^2\left(\frac{\pi}{2k}\right)}\sin^2\left(\frac{\pi}{k}\right)}{\sin^2\left(\frac{\pi}{k}\right)} = 2\cos\left(\frac{\pi}{2k}\right)\sqrt{1+\cos^2\left(\frac{\pi}{2k}\right)} < 2\sqrt{2}.$$

■

Now we can assume that $\alpha_{12} \geq \pi/5$. Only one or two angles among remaining α_{ij} can be equal to π/k, where $k \geq 6$. Without loss of generality, we suppose that $\alpha_{13} = \frac{\pi}{k}$, where $k \geq 6$. Then $\alpha_{12} = \alpha_{23} = \pi/2$.

If no α_{ij} is equal to π/k for $k \geq 6$, then these angles can equal only $\pi/2$, $\pi/3$, $\pi/4$, and $\pi/5$. Recall that any triple of angles around one of the vertices of the edge E contains $\pi/2$.

Thus, we can consider separately the case $\alpha_{12} = \pi/2$.

Proposition 3.2. *If $\alpha_{12} = \pi/2$, then $\mathbf{t}_{\overline{\alpha}} < 5$.*

Proof. We have $\overline{\alpha} = \left(\frac{\pi}{2}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}\right)$. Let us now compute:

$$\sqrt{G_{33}G_{44}} = \sqrt{1-\cos^2\alpha_{13}-\cos^2\alpha_{23}}\sqrt{1-\cos^2\alpha_{14}-\cos^2\alpha_{24}}. \tag{2}$$

Notice that (by Lemma 2.1)

$$\cos\alpha_1 = \frac{\cos\alpha_{23}}{\sin\alpha_{13}}, \qquad \cos\alpha_2 = \frac{\cos\alpha_{24}}{\sin\alpha_{14}},$$
$$\cos\alpha_3 = \frac{\cos\alpha_{13}}{\sin\alpha_{23}}, \qquad \cos\alpha_4 = \frac{\cos\alpha_{14}}{\sin\alpha_{24}}.$$

Using the above expressions and Lemma 2.2, we have

$$\begin{aligned}
&\mathbf{t}_{\overline{\alpha}} \leq \cosh F_{1,2}(\overline{\alpha})\sqrt{\sin^2\alpha_{13}-\cos^2\alpha_{23}}\sqrt{\sin^2\alpha_{14}-\cos^2\alpha_{24}}\\
&\leq \frac{\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right)}{\sqrt{1-\cos\alpha_1}\sqrt{1-\cos\alpha_2}}\sqrt{\sin^2\alpha_{13}-\cos^2\alpha_{23}}\sqrt{\sin^2\alpha_{14}-\cos^2\alpha_{24}} +\\
&+\frac{\sqrt{1+\sin^2\left(\frac{\alpha_1}{2}\right)}\sqrt{1+\sin^2\left(\frac{\alpha_2}{2}\right)}}{\sqrt{1-\cos\alpha_1}\sqrt{1-\cos\alpha_2}}\sqrt{\sin^2\alpha_{13}-\cos^2\alpha_{23}}\sqrt{\sin^2\alpha_{14}-\cos^2\alpha_{24}} \leq\\
&\leq \frac{\cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right)\sqrt{\sin\alpha_{13}\sin\alpha_{14}}}{\sqrt{\sin\alpha_{13}-\cos\alpha_{23}}\sqrt{\sin\alpha_{14}-\cos\alpha_{24}}}\sqrt{\sin^2\alpha_{13}-\cos^2\alpha_{23}}\sqrt{\sin^2\alpha_{14}-\cos^2\alpha_{24}} +\\
&+\frac{\sqrt{1+\sin^2\left(\frac{\alpha_1}{2}\right)}\sqrt{1+\sin^2\left(\frac{\alpha_2}{2}\right)}\sqrt{\sin\alpha_{13}\sin\alpha_{14}}}{\sqrt{\sin\alpha_{13}-\cos\alpha_{23}}\sqrt{\sin\alpha_{14}-\cos\alpha_{24}}}\sqrt{\sin^2\alpha_{13}-\cos^2\alpha_{23}}\sqrt{\sin^2\alpha_{14}-\cos^2\alpha_{24}} \leq\\
&\leq \cos\left(\frac{\alpha_1}{2}\right)\cos\left(\frac{\alpha_2}{2}\right)\sqrt{\sin\alpha_{13}+\cos\alpha_{23}}\sqrt{\sin\alpha_{14}+\cos\alpha_{24}} +\\
&+\sqrt{1+\sin^2\left(\frac{\alpha_1}{2}\right)}\sqrt{1+\sin^2\left(\frac{\alpha_2}{2}\right)}\sqrt{\sin\alpha_{13}+\cos\alpha_{23}}\sqrt{\sin\alpha_{14}+\cos\alpha_{24}} <\\
&< \sqrt{2}\sqrt{2} + \frac{\sqrt{3}}{\sqrt{2}}\frac{\sqrt{3}}{\sqrt{2}}\sqrt{2}\sqrt{2} = 5.
\end{aligned}$$

The last inequality holds since $\alpha_i \leq \pi/2$, and, therefore, $\sin(\alpha_i/2) \leq \sin(\pi/4) = 1/\sqrt{2}$. Absolutely the same argument works for $\mathbf{t}_{\overline{\alpha}}$ bounded via $\cosh F_{3,4}(\overline{\alpha})$. ■

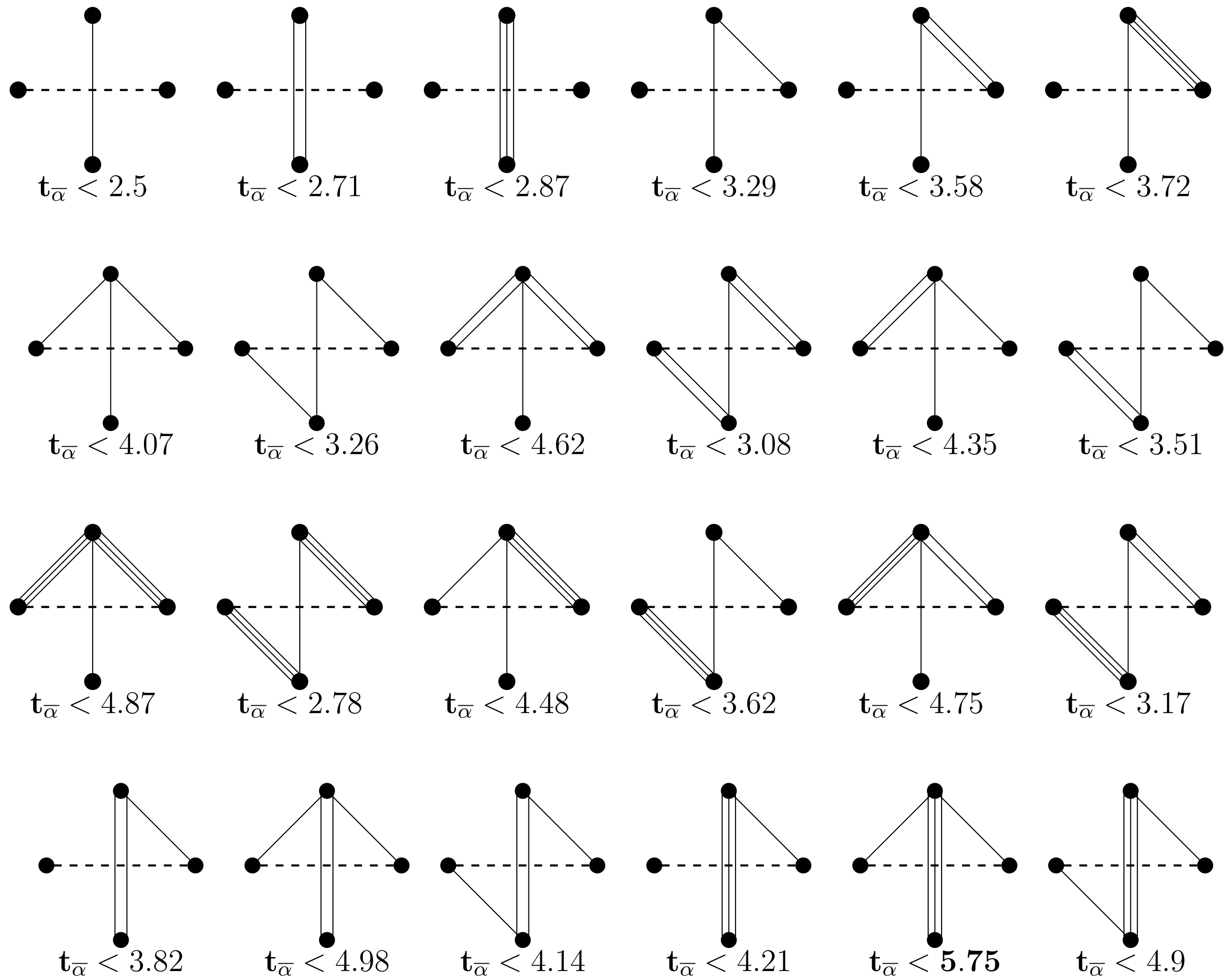

Table 1. Coxeter diagrams of the remaining small ridges

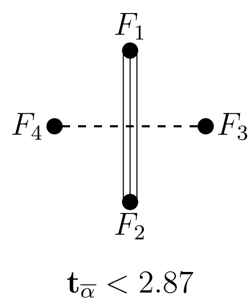

Figure 2. Coxeter diagtam of a small ridge $(\pi/5, \pi/2, \pi/2, \pi/2, \pi/2)$.

After that, it remains to calculate $\mathbf{t}_{\overline{\alpha}}$ for 24 different types of a small ridge. It was done using the program `SmaRBA` [11] written in a computer algebra system `Sage`. The result is presented in Table 1 as the list of Coxeter diagrams for facets F_1, F_2, F_3, F_4. The facets F_3 and F_4 will be connected by a dotted line, and the whole diagram will be signed by the relevant bound: $\mathbf{t}_{\overline{\alpha}} <$ constant. The numbering of the facets of each diagram is the same as in Fig. 2, which shows an example of what the ridge diagram looks like when $\overline{\alpha} = (\pi/5, \pi/2, \pi/2, \pi/2, \pi/2)$. We see from this picture that $\mathbf{t}_{(\pi/5,\pi/2,\pi/2,\pi/2,\pi/2)} < 2.87$.

The numbers given in Table 1 were calculated by `SmaRBA` [11] with accuracy up to eight decimal places. We show them rounded up to the nearest hundredth, which is quite enough for our purposes. For example, the maximal found number approximately equals 5.74850431686, which was rounded up to 5.75.

Thus, combining Propositions 3.1, 3.2, and this list in Table 1, we obtain the proof of Theorem A. □

3.3. Proof of Corollary 1. Let P be a compact Coxeter polytope in $\mathbb{H}^{n\geq 4}$. Suppose that P' is a 3-dimensional face of P, which is itself a Coxeter polytope. Let O be the interior point of P', and $E \in P'$ be the outermost edge from this point.

Then P' has (2-dimensional) facets F_1 and F_2, framing the edge E, and, by Theorem A, $\cosh \rho(F_1, F_2) \leq \mathbf{t}_{\overline{\alpha}} < 5.75$. Recall that a compact hyperbolic Coxeter polytope P is simple. This implies that F_1 and F_2 belong to facets P_1 and P_2 of P, respectively, where P_1 and P_2 are also the framing facets for the edge E. Then we have

$$\cosh \rho(P_1, P_2) \leq \cosh \rho(F_1, F_2) \leq \mathbf{t}_{\overline{\alpha}} < 5.75.$$

□

§ 4. Proof of Theorem B

The distance from the point $e_0 \in \mathbb{H}^n$, where $(e_0, e_0) = -1$, to the plane

$$H_{u_1,\ldots,u_k} := \{x \in \mathbb{H}^n \mid x \in \langle u_1, \ldots, u_k\rangle^{\perp},\ (u_j, u_j) = 1,\ 1 \leq j \leq k\}$$

can be calculated by the formula

$$\sinh^2 \rho(e_0, H_{u_1,\ldots,u_k}) = \sum_{i,j} \overline{g_{ij}} y_i y_j, \tag{3}$$

where $\overline{g_{ij}}$ are the elements of the inverse matrix $G^{-1} = G(u_1, \ldots, u_k)^{-1}$, and

$$y_j = -(e_0, u_j) = -\sinh \rho(e_0, H_j)$$

for all $1 \leq j \leq k$ (we assume that $(e_0, u_j) \leq 0$, i.e. $e_0 \in H^-_{u_j}$).

Let P be a compact Coxeter polytope in $\mathbb{H}^n$ whose small ridge (associated with the outermost edge E from some point $O \in P$ given by the vector $e_0 \in \mathbb{E}^{n,1}$ such that $(e_0, e_0) = -1$) is right-angled. Let $F_1, \ldots, F_{n-1}$ be the facets of P containing E with unit outer normals $u_1, \ldots, u_{n-1}$, and let u_n and u_{n+1} be the unit outer normals to the framing facets F_n and F_{n+1} containing the vertices of E but not E itself.

Let us consider the following Gram matrix

$$G(e_0, u_1, u_2, \ldots, u_{n+1}) = \begin{pmatrix} -1 & -y_1 & \ldots & -y_n & -y_{n+1} \\ -y_1 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -y_n & 0 & 0 & 1 & -T \\ -y_{n+1} & 0 & 0 & -T & 1 \end{pmatrix},$$

where

$$y_j = -(e_0, u_j) = -\sinh \rho(e_0, H_j), \quad H_j = \{x \mid (x, u_j) = 0\}.$$

We can assume that $y_n \leq y_{n+1}$. The fact that the edge E is the outermost edge from the point O, gives us the inequalities $\rho(O, E) \geq \rho(O, E')$ for any edge E' adjacent to E. Recall that $E' \in H_n$ or $E' \in H_{n+1}$. Assume that $E' \in H_n$ and $E' \notin H_j$. By (3), the distances from the point O to edges E and E' of P satisfy the following:

$$\sinh^2 \rho(O,E) = y_1^2 + \ldots y_{n-1}^2, \quad \sinh^2 \rho(O,E') = y_1^2 + \ldots + y_{j-1}^2 + y_{j+1}^2 + \ldots + y_{n-1}^2 + y_n^2.$$

Applying the above consideration to every edge E' adjacent to E, we obtain that

$$y_n \le y_{n+1} \le y_1, y_2, \ldots, y_{n-1}.$$

Since $n+2$ vectors $e_0, e_1, \ldots, e_{n+1}$ belong to $(n+1)$-dimensional vector space $\mathbb{E}^{n,1}$, then

$$\det G(e_0, e_1, \ldots, e_{n+1}) = (y_1^2 + \ldots + y_{n-1}^2 + 1)T^2 - 2y_n y_{n+1} T - (y_1^2 + \ldots + y_{n+1}^2 + 1) = 0,$$

i.e.,

$$T = \frac{y_n y_{n+1} + \sqrt{y_n^2 y_{n+1}^2 + AB}}{A},$$

where

$$A := y_1^2 + \ldots + y_{n-1}^2 + 1, \quad B := y_1^2 + \ldots + y_{n+1}^2 + 1.$$

Therefore,

$$2y_n y_{n+1} \le y_n^2 + y_{n+1}^2 < A \le B, \quad B/A = 1 + y_n y_{n+1}/A < 1.5$$

and

$$T < 0.5 + \sqrt{0.25 + 1 + 1} = 2.$$

□

§ 5. Arithmetic hyperbolic reflection groups and reflective Lorentzian lattices

5.1. Definitions and preliminaries. Suppose that $\mathbb{F}$ is a totally real algebraic number field with the ring of integers $A = \mathcal{O}_{\mathbb{F}}$. For convenience we will assume that it is a principal ideal domain.

Definition 5.1. *A free finitely generated A-module L with an inner product of signature $(n,1)$ is said to be a Lorentzian lattice if, for each non-identity embedding $\sigma\colon \mathbb{F} \to \mathbb{R}$, the quadratic space $L \otimes_{\sigma(A)} \mathbb{R}$ is positive definite.*

Suppose that L is a Lorentzian lattice. It is embedded in the $(n+1)$-dimensional real *Minkowski space* $\mathbb{E}^{n,1} = L \otimes_{\mathrm{id}(A)} \mathbb{R}$. We shall take one of the connected components of the hyperboloid

$$\{v \in \mathbb{E}^{n,1} \mid (v,v) = -1\} \tag{4}$$

as a vector model of the n-dimensional hyperbolic Lobachevsky space $\mathbb{H}^n$.

Suppose that $\mathcal{O}(L)$ is the group of automorphisms of a lattice L. It is known (cf. [38, 12, 25]) that its subgroup $\mathcal{O}'(L)$ leaving invariant each connected component of the hyperboloid (4), is a discrete group of motions of the Lobachevsky space with finite volume fundamental polytope. Moreover, if $\mathbb{F} = \mathbb{Q}$ and the lattice L is isotropic (that is, the quadratic form associated with it represents zero), then the quotient space $\mathbb{H}^n/\Gamma$ is a finite volume non-compact orbifold, and in all other cases it is compact.

Definition 5.2. *The groups Γ obtained in the above way and the subgroups of the group* $\mathrm{Isom}(\mathbb{H}^n)$ *that are commensurable*[6] *with them are called arithmetic discrete groups (or lattices) of the simplest type. The field $\mathbb{F}$ is called the field of definition (or the ground field) of the group Γ (and all subgroups commensurable with it).*

[6]Two subgroups Γ_1 and Γ_2 of some group are said to be commensurable if the group $\Gamma_1 \cap \Gamma_2$ is a subgroup of finite index in each of them.

A primitive vector e of a Lorentzian lattice L is called a *root* or, more precisely, a *k-root*, where $k = (e,e) \in A_{>0}$ if $2(e,x) \in kA$ for all $x \in L$. Every root e defines an *orthogonal reflection* (called a *k-reflection* if $(e,e) = k$) in the space $L \otimes_{\mathrm{id}(A)} \mathbb{R}$

$$\mathcal{R}_e : x \mapsto x - \frac{2(e,x)}{(e,e)} e,$$

which preserves the lattice L and determines the reflection of the space $\mathbb{H}^n$ with respect to the hyperplane $H_e = \{x \in \mathbb{H}^n \mid (x,e) = 0\}$, called the *mirror* of $\mathcal{R}_e$.

Definition 5.3. *A reflection $\mathcal{R}_e$ is called stable if $(e,e) \mid 2$ in A.*

For example, for $\mathbb{F} = \mathbb{Q}$ and $A = \mathbb{Z}$, this holds for $(e,e) = 1$ and $(e,e) = 2$, i.e., only 1- and 2-reflections are stable, while, for $\mathbb{F} = \mathbb{Q}[\sqrt{2}]$ and $A = \mathbb{Z}[\sqrt{2}]$, stable are 1-, 2-, and $(2+\sqrt{2})$-reflections. Any primitive vector $e \in L$ for which $(e,e) \mid 2$ is automatically a root of the lattice L and of any of its finite extensions.

Let L be a Lorentzian lattice over a ring of integers A. We denote by $\mathcal{O}_r(L)$ the subgroup of the group $\mathcal{O}'(L)$ generated by all reflections contained in it, and we denote by $S(L)$ the subgroup of $\mathcal{O}'(L)$ generated by all stable reflections.

Definition 5.4. *A Lorentzian lattice L is said to be reflective if the index $[\mathcal{O}'(L) : \mathcal{O}_r(L)]$ is finite, and stably reflective if the index $[\mathcal{O}(L) : S(L)]$ is finite.*

Remark 4. *In [5, 6, 8] stably reflective lattices over $\mathbb{Z}$ are called (1,2)-reflective.*

Definition 5.5. *A Lorentzian $\mathbb{Z}$-lattice L is called 2-reflective if the subgroup $\mathcal{O}_r^{(2)}(L)$ generated by all 2-reflections has a finite index in $\mathcal{O}'(L)$.*

Note that any 2-reflective lattice is stably reflective. Obviously, a finite extension of any stably reflective Lorentzian lattice is also a stably reflective Lorentzian lattice.

5.2. State of the art. As mentioned in the introduction, Vinberg [39] started in 1967 a systematic study of hyperbolic reflection groups. He proved an arithmeticity criterion for finite covolume hyperbolic reflection groups and, in particular, he showed that a discrete hyperbolic reflection group of finite covolume is an arithmetic group with ground field $\mathbb{F}$ if and only if it is commensurable with a group of the form $\mathcal{O}'(L)$, where L is some (automatically reflective) Lorentzian lattice over a totally real number field $\mathbb{F}$.

In 1972, Vinberg proposed an algorithm (see [40], [42]) that, given a lattice L, enables one to construct the fundamental Coxeter polytope of the group $\mathcal{O}_r(L)$ and determine thereby the reflectivity of the lattice L.

The next important result belongs to several authors.

Theorem 5.1 (see [44, 28, 22, 1, 32, 2])**.** *For each $n \geq 2$, up to scaling, there are only finitely many reflective Lorentzian lattices of signature $(n,1)$. Similarly, up to conjugacy, there are only finitely many maximal arithmetic reflection groups in the spaces $\mathbb{H}^n$. Arithmetic hyperbolic reflection groups and compact Coxeter polytopes do not exist in $\mathbb{H}^{n \geq 30}$.*

It was also proved that there are no reflective hyperbolic $\mathbb{Z}$-lattices of rank $n+1 > 22$ (F. Esselmann, 1996 [17]).

The above results give the hope that all reflective Lorentzian lattices, as well as maximal arithmetic hyperbolic reflection groups can be classified.

Here we describe a progress in the problem of classification of reflective Lorentzian lattices. A more detailed history of the problem can be found in recent survey of Belolipetsky [4].

For the ground field $\mathbb{Q}$: the reflective Loretzian lattices of signature $(n, 1)$ are classified for $n = 2$ (V.V. Nikulin, 2000 [31], and D. Allcock, 2011 [3]), $n = 4$ (R. Sharlau and C. Walhorn, 1989–1993 [36, 51]), $n = 5$ (I. Turkalj, 2017 [37]) and in the non-compact (isotropic) case for $n = 3$ (R. Sharlau and C. Walhorn, 1989–1993 [35, 36]).

A classification of reflective Lorentzian lattices of signature $(2, 1)$ over $\mathbb{Z}[\sqrt{2}]$ was obtained by A. Mark in 2015 [23, 24].

Unimodular reflective Lorentzian lattices over $\mathbb{Z}$ were classified by Vinberg and Kaplinskaja, (1972 and 1978, see [40, 41, 43]). Other classifications of unimodular reflective Lorentzian lattices over $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[(1+\sqrt{5})/2]$ and $\mathbb{Z}[\cos(2\pi/7)]$, were obtained by Bugaenko (1984, 1990 and 1992, see [13, 14, 15]).

In 1979, 1981, and 1984 (cf. [26, 28, 30]), Nikulin obtained classification of 2-reflective Lorentzian $\mathbb{Z}$-lattices of signature $(n, 1)$ for $n \neq 3$, and Vinberg classified these lattices for $n = 3$ (cf. 1998 and 2007, [46, 47]). Finally, the author of this paper obtained (cf. [5, 6, 8]) a classification of stably reflective anisotropic Lorentzian $\mathbb{Z}$-lattices of signature $(3, 1)$. (They all turned out to be 2-reflective in this case.)

In all other cases, the classification problem still remains open.

5.3. Methods of testing a lattice for stable reflectivity and non-reflectivity. Recall that there is Vinberg's algorithm that constructs the fundamental Coxeter polytope of the group $\mathcal{O}_r(L)$. It can be applied to the group of type $S(L)$. However, it will be more efficient to apply the procedure of Vinberg's algorithm to the large group $\mathcal{O}_r(L)$ and to use some another approach to determine whether L is stably reflective or not.

5.3.1. *The method of "bad" reflections.* If we can construct the fundamental Coxeter polyhedron (or some part of it) of the group $\mathcal{O}_r(L)$ for some Lorentzian lattice L, then it is possible to determine whether it is stably reflective. One can consider the group Δ generated by the k-reflections that are not stable (we shall call them "bad" reflections) in the sides of the fundamental polyhedron of the group $\mathcal{O}_r(L)$. The following lemma holds (see [47]).

Lemma 5.1. *A lattice L is stably reflective if and only if it is reflective and the group Δ is finite.*

Actually, to prove that a lattice is not stably reflective, it is sufficient to construct only some part of the fundamental polyhedron containing an infinite subgroup generated by bad reflections.

5.3.2. *Method of infinite symmetry.* Recall that

$$\mathcal{O}'(L) = \mathcal{O}_r(L) \rtimes H,$$

where $H = \mathrm{Sym}(P) \cap \mathcal{O}'(L)$. If P is of infinite volume and has infinitely many faces, then the group H is infinite. To determine whether it is infinite or not, one can use the following lemma proved by V. O. Bugaenko in 1992 (see [15]).

Lemma 5.2. *Suppose H is a discrete subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$. *Then H is infinite if and only if there exists a subgroup of H without fixed points in $\mathbb{H}^n$.*

How can we find the set of fixed points?

$\mathbb{F}$	Possible values for (u,u)	Possible angles	# of different ridges	max $\mathbf{t}_{\overline{\alpha}}$
$\mathbb{Q}$	1, 2	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$	44	4.98
$\mathbb{Q}[\sqrt{2}]$	1, 2, $2+\sqrt{2}$	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{8}$	58	4.98
$\mathbb{Q}[\sqrt{3}]$	1, 2, $2+\sqrt{3}$	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{12}$	58	4.98
$\mathbb{Q}[\sqrt{5}]$	1, 2	$\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{10}$	99	5.75

TABLE 2. Some quantities for stably reflective Lorentzian lattices over ground fields $\mathbb{F} = \mathbb{Q}, \mathbb{Q}[\sqrt{2}], \mathbb{Q}[\sqrt{3}], \mathbb{Q}[\sqrt{5}]$, i.e. $\mathcal{O}_{\mathbb{F}} = \mathbb{Z}, \mathbb{Z}[\sqrt{2}], \mathbb{Z}[\sqrt{3}], \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$.

Lemma 5.3 (Bugaenko, see Lemma 3.2 in [15]). *Let η be an involutive trasnformation of a real vector space V. Then the set of its fixed points $Fix(\eta)$ is generated by vectors $e_j + \eta(e_j)$, where $\{e_j\}$ form a basis of V.*

Due to this lemma the proof of non-reflectivity of a lattice is the following. If we know a part of a polyhedron P for the group $\mathcal{O}_r(L)$, then we can find a few symmetries of its Coxeter diagram.

If these symmetries preserve the lattice L, then they generate the subgroup that preserves P. If this subgroup has no fixed points then $\mathcal{O}_r(L)$ is of infinite index in $\mathcal{O}'(L)$.

§ 6. Classification of stably reflective Lorentzian lattices of signature $(3,1)$

6.1. Description of the method. In this section we describe application of Theorem A or, more precisely, of Theorem 3.1 to classification of stably reflective Lorentzian lattices.

Let now P be the fundamental Coxeter polytope of the group $\mathcal{O}_r(L)$ for an anisotropic Lorentzian lattice L of signature $(3,1)$ over a ring of integers $\mathcal{O}_F$ of any totally real number field $\mathbb{F}$. The lattice L is reflective if and only if the polytope P is compact (i. e., bounded) in $\mathbb{H}^3$.

Let E be an edge (of the polytope P) corresponding to a small ridge of width not greater than $\mathbf{t}_{\overline{\alpha}}$. By Theorem A we can ensure that $\mathbf{t}_{\overline{\alpha}} < 5.75$, however, we shall use a more efficient way, an explicit formula from Theorem 3.1.

Indeed, for a fixed number field $\mathbb{F}$ only finitely many dihedral angles in Coxeter polytopes are possible. This leaves us only finitely many combinatorial types of a small ridge, and for each such type one can explicitly compute (see `SmaRBA` [11]) the respective bound $\mathbf{t}_{\overline{\alpha}}$. We present some useful calculations in Table 2.

Remark 5. *Due to some technical mistake, the bound $\mathbf{t}_{\overline{\alpha}} < 4.14$ in* [8, Theorem 1.1] *is incorrect (the correct one is $\mathbf{t}_{\overline{\alpha}} < 4.98$). However the result* [8, Theorem 1.2] *is still correct.*

Let u_1, u_2 be the roots of the lattice L that are orthogonal to the facets containing the edge E and are the outer normals of these facets. Similarly, let u_3, u_4 be the roots corresponding to the framing facets. We denote these facets by F_1, F_2, F_3, and F_4, respectively. If $(u_3, u_3) = k$, $(u_4, u_4) = l$, then (by Theorem A)

$$|(u_3, u_4)| \leq 5.75\sqrt{kl}. \tag{5}$$

Since we are solving the classification problem for stably reflective lattices, all roots of L satisfy the condition $(u,u) \mid 2$ in $\mathcal{O}_F$. Thus, (u,u) always takes finitely many values (see Table 2).

In this case we are given bounds on all elements of the matrix $G(u_1, u_2, u_3, u_4)$, because all the facets F_i are pairwise intersecting, excepting, possibly, the pair of faces F_3 and F_4. But

Table 3. Unimodular reflective Lorentzian lattices over $\mathbb{Q}[\sqrt{13}]$ and $\mathbb{Q}[\sqrt{17}]$.

L	n	# facets
$[-\frac{3+\sqrt{13}}{2}] \oplus [1] \oplus \ldots \oplus [1]$	2	4
$[-\frac{3+\sqrt{13}}{2}] \oplus [1] \oplus \ldots \oplus [1]$	3	9
$[-\frac{3+\sqrt{13}}{2}] \oplus [1] \oplus \ldots \oplus [1]$	4	40

L	n	# facets
$[-4-\sqrt{17}] \oplus [1] \oplus \ldots \oplus [1]$	2	4
$[-4-\sqrt{17}] \oplus [1] \oplus \ldots \oplus [1]$	3	6
$[-4-\sqrt{17}] \oplus [1] \oplus \ldots \oplus [1]$	4	20

if they do not intersect, then the distance between these faces is bounded by inequality (5). Thus, all entries of the matrix $G(u_1, u_2, u_3, u_4)$ are integer and bounded, so there are only finitely many possible matrices $G(u_1, u_2, u_3, u_4)$.

The vectors u_1, u_2, u_3, u_4 generate some sublattice L' of finite index in the lattice L. More precisely, the lattice L lies between the lattices L' and $(L')^*$, and

$$[(L')^* : L']^2 = |d(L')|.$$

Hence it follows that $|d(L')|$ is divisible by $[L : L']^2$. Using this, in each case we shall find for a lattice L' all its possible extensions of finite index.

The resulting list of candidate lattices is verified for reflectivity using Vinberg's algorithm. There exist a few software implementations of Vinberg's algorithms, these are `AlVin` [18, 19], for Lorentzian lattices with an orthogonal basis over several ground fields, and `VinAl` (cf. [50, 7]) for Lorentzian lattices with an arbitrary basis over $\mathbb{Z}$. Further work on the project that implements Vinberg's algorithm for arbitrary lattices over the quadratic fields $\mathbb{Z}[\sqrt{d}]$ is being carried out jointly with A. Yu. Perepechko.

We introduce some notation:

1) $[C]$ is a quadratic lattice whose inner product in some basis is given by a symmetric matrix C;
2) $d(L) := \det C$ is the discriminant of the lattice $L = [C]$;
3) $L \oplus M$ is the orthogonal sum of the lattices L and M.

The method described above, allows one to obtain the following two facts.

Theorem 6.1 (Bogachev, [8], Th. 1.2). *Every stably reflective anisotropic Lorentzian lattice of signature* $(3,1)$ *over* $\mathbb{Z}$ *is either isomorphic to* $[-7] \oplus [1] \oplus [1] \oplus [1]$ *or* $[-15] \oplus [1] \oplus [1] \oplus [1]$*, or to an even index* 2 *sublattice of one of them.*

Using `AlVin`, one can easy get the following.

Theorem 6.2. *Unimodular Lorentzian lattices over* $\mathbb{Q}[\sqrt{13}]$ *and* $\mathbb{Q}[\sqrt{17}]$ *of signature* $(n,1)$ *are reflective if and only if* $n \leq 4$ *(see Table 3 for details).*

The next step is to find a short list of candidate-lattices for stable reflectivity.

6.2. Short list of candidate-lattices. Our program `SmaRBA` [11] creates a list of numbers $\mathbf{t}_{\overline{\alpha}}$ (with respect to the ground field $\mathbb{Q}[\sqrt{2}]$, see Table 2) and then, using this list, displays all Gram matrices $G(u_1, u_2, u_3, u_4)$.

This list consists of 83 matrices, but many of them are pairwise isomorphic. After the restriction of this list, we obtain matrices G_1–G_{15}, for each of which we find all corresponding extensions.

To each Gram matrix G_k in our notation, there corresponds a lattice L_k that can have some other extensions. For each new lattice (non-isomorphic to any previously found lattice) we introduce the notation $L(k)$, where k denotes its number:

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1-\sqrt{2} \\ 0 & 0 & -1-\sqrt{2} & 1 \end{pmatrix}, \qquad L_1 \simeq [-2(1+\sqrt{2})] \oplus [1] \oplus [1] \oplus [1];$$

Its unique extension is an "index $\sqrt{2}$" extension

$$L(1) := [-(1+\sqrt{2})] \oplus [1] \oplus [1] \oplus [1].$$

$$G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1-\sqrt{2} \\ 0 & 0 & -1-\sqrt{2} & 2 \end{pmatrix}, \qquad L_2 \simeq [-(1+2\sqrt{2})] \oplus [1] \oplus [1] \oplus [1] := L(2);$$

$$G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1-\sqrt{2} \\ 0 & 0 & -1-\sqrt{2} & 2 \end{pmatrix}, \qquad L_3 \simeq \begin{bmatrix} 2 & -1-\sqrt{2} \\ -1-\sqrt{2} & 2 \end{bmatrix} \oplus [1] \oplus [1] := L(3);$$

$$G_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1-2\sqrt{2} \\ 0 & 0 & -1-2\sqrt{2} & 2 \end{pmatrix}, \; L_4 \simeq \begin{bmatrix} 2 & -1-2\sqrt{2} \\ -1-2\sqrt{2} & 2 \end{bmatrix} \oplus [1] \oplus [1] := L(4);$$

$$G_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 2 & -1-\sqrt{2} \\ 0 & -1 & -1-\sqrt{2} & 2 \end{pmatrix}, \; L_5 \simeq [-5-4\sqrt{2}] \oplus [1] \oplus [1] \oplus [1] := L(5);$$

$$G_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 2 & -1-2\sqrt{2} \\ 0 & -1 & -1-2\sqrt{2} & 2 \end{pmatrix}, \; L_6 \simeq [-11-8\sqrt{2}] \oplus [1] \oplus [1] \oplus [1] := L(6);$$

$$G_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2-2\sqrt{2} \\ 0 & 0 & -2-2\sqrt{2} & 2 \end{pmatrix}, \; L_7 = [G_7];$$

Its unique extension is an "index $\sqrt{2}$" extension

$$L(7) := [-\sqrt{2}] \oplus [1] \oplus [1] \oplus [1].$$

$$G_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2} \\ 0 & -1 & -\sqrt{2} & 2 \end{pmatrix}, \; L_8 = [G_8];$$

Its unique extension is an "index $\sqrt{2}$" extension

$$L(8) := \begin{bmatrix} 2 & -1 & -\sqrt{2} \\ -1 & 2 & \sqrt{2}-1 \\ -\sqrt{2} & \sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix} \oplus [1].$$

$$G_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2}-1 \\ 0 & -1 & -\sqrt{2}-1 & 2 \end{pmatrix}, \ L_9 = [G_9];$$

Its unique extension is an "index $\sqrt{2}$" extension

$$L(9) := \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -\sqrt{2} \end{bmatrix} \oplus [1].$$

$$G_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2}-2 \\ 0 & -1 & -\sqrt{2}-2 & 2 \end{pmatrix}, \ L_{10} = [G_{10}];$$

Its unique extension is an "index $\sqrt{2}$" extension

$$L(10) := \begin{bmatrix} 2 & -1-\sqrt{2} \\ -1-\sqrt{2} & 2 \end{bmatrix} \oplus [2+\sqrt{2}] \oplus [1].$$

$$G_{11} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2}-1 \\ -1 & -1 & -\sqrt{2}-1 & 2 \end{pmatrix}, \ L_{11} \simeq [-7-6\sqrt{2}] \oplus [1] \oplus [1] \oplus [1] := L(11);$$

$$G_{12} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & -2\sqrt{2}-1 \\ -1 & -1 & -2\sqrt{2}-1 & 2 \end{pmatrix}, \ L_{12} = [G_{12}];$$

Its unique extension is an index 2 extension

$$L(12) := [-7-5\sqrt{2}] \oplus [1] \oplus [1] \oplus [1].$$

$$G_{13} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2} \\ -1 & -1 & -\sqrt{2} & 2 \end{pmatrix}, \ L_{13} = [G_{13}] := L(13);$$

$$G_{14} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -\sqrt{2}-1 \\ -1 & -1 & -\sqrt{2}-1 & 2 \end{pmatrix}, \ L_{14} = [G_{14}] := L(14);$$

$$G_{15} = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -\sqrt{2} \\ -1 & -1 & 2 & -\sqrt{2}-1 \\ -1 & -\sqrt{2} & -\sqrt{2}-1 & 2 \end{pmatrix}, \ L_{15} = [G_{15}] := L(15).$$

§ 7. Stable reflectivity test and proof of Theorem C

So far we have 15 candidate lattices $L(1)$–$L(15)$. For each lattice $L(k)$ we will use Vinberg's algorithm for constructing the fundamental Coxeter polytope for the group $\mathcal{O}_r(L(k))$. After that, it remains to apply Lemma 5.1.

First of all, we study candidate lattices with an orthogonal basis. We apply software implementation `AlVin` [18] of Vinberg's algorithm. This program is written for Lorentzian lattices associated with diagonal quadratic forms with square-free coefficients.

For lattices with non-orthogonal basis we use another approach. For every such lattice L, we find a sublattice $L' \subset \mathbb{Q}^4[\sqrt{2}]$ isomorphic to L and given by an inner product associated with a diagonal quadratic form. Further, our program `VinAl` [50] finds roots of L'.

As the result, we obtain seven stably reflective Lorentzian lattices of signature $(3,1)$ over $\mathbb{Z}[\sqrt{2}]$, which are represented in Table 4.

Remark 6. *The lattice* $L(8) = \begin{bmatrix} 2 & -1 & -\sqrt{2} \\ -1 & 2 & \sqrt{2}-1 \\ -\sqrt{2} & \sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix} \oplus [1]$ *is isomorphic to the lattice with coordinates* $y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$ *and with inner product given by the quadratic form* $f(y) = -\sqrt{2}\, y_0^2 + y_1^2 + y_2^2 + y_3^2$, *where* $y_0 \in \mathbb{Z}[\sqrt{2}]$, $-y_2 + \frac{y_1+y_2}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}]$, $\sqrt{2}y_2, y_3 \in \mathbb{Z}[\sqrt{2}]$. *The roots of* $L(8)$ *in Table 4 are given in these* ***new*** *coordinates.*

The Gram matrices and Coxeter diagrams corresponding to all the lattices $L(1)$–$L(15)$ can be obtained by `SmaRBA` [11].

We shall prove that all remaining lattices are not stably reflective (some of them are reflective but not stably reflective).

Proposition 7.1. *The lattice* $L(11) = [-7-6\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$ *is reflective, but not stably reflective.*

Proof. For the lattice $L(11)$ we apply Vinberg's algorithm. The program `AlVin` [18] found 10 roots:

$$a_1 = (0,\ -1,\ 1,\ 0)\,, \qquad a_2 = (0,\ 0,\ -1,\ 1)\,,$$
$$a_3 = (0,\ 0,\ 0,\ -1)\,, \qquad a_4 = \left(1,\ \sqrt{2}+1,\ \sqrt{2}+1,\ \sqrt{2}+1\right),$$
$$a_5 = \left(1,\ \sqrt{2}+2,\ \sqrt{2}+1,\ 0\right), \qquad a_6 = \left(2\sqrt{2}+1,\ 6\sqrt{2}+7,\ 0,\ 0\right),$$
$$a_7 = \left(\sqrt{2}+1,\ 3\sqrt{2}+5,\ \sqrt{2}+1,\ 1\right), \qquad a_8 = \left(\sqrt{2}+1,\ 3\sqrt{2}+4,\ \sqrt{2}+2,\ \sqrt{2}+2\right),$$
$$a_9 = \left(4\sqrt{2}+6,\ 13\sqrt{2}+19,\ 7\sqrt{2}+12,\ 6\sqrt{2}+7\right), \quad a_{10} = \left(2\sqrt{2}+2,\ 6\sqrt{2}+9,\ 2\sqrt{2}+3,\ 2\sqrt{2}+2\right).$$

The Gram matrix of this set of roots corresponds to a compact 3-dimensional Coxeter polytope. The main diagonal of this matrix equals

$$\{2, 2, 1, 2, 2, 2\sqrt{2}+10, 2, 1, 2\sqrt{2}+10\}.$$

It remains to see that the group generated by "bad" reflections with respect to mirrors H_{a_6} and $H_{a_{10}}$, is infinite, since the respective vertices of the Coxeter diagram are connected by the dotted edged. Hence, the lattice $L(11)$ is reflective, but not stably reflective. ∎

Proposition 7.2. *The lattice* $L(3) = \begin{bmatrix} 2 & -1-\sqrt{2} \\ -1-\sqrt{2} & 2 \end{bmatrix} \oplus [1] \oplus [1]$ *is reflective, but not stably reflective.*

$L(k)$	L	Roots	$B(L)$
$L(1)$	$[-1-\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	$a_1 = (0,-1,1,0) \quad a_2 = (0,0,-1,1)$ $a_3 = (0,0,0,-1) \quad a_4 = \left(1, 1+\sqrt{2}, 0, 0\right)$ $a_5 = \left(1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\right)$	$\emptyset$
$L(2)$	$[-1-2\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	$a_1 = (0,-1,1,0) \quad a_2 = (0,0,-1,1)$ $a_3 = (0,0,0,-1) \quad a_4 = \left(1, 1+\sqrt{2}, 0, 0\right)$ $a_5 = \left(1+\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, 1\right)$ $a_6 = \left(1+\sqrt{2}, 2+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\right)$	$\emptyset$
$L(5)$	$[-5-4\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	$a_1 = (0,-1,1,0) \quad a_2 = (0,0,-1,1)$ $a_3 = (0,0,0,-1) \quad a_4 = \left(1, 3+\sqrt{2}, 0, 0\right)$ $a_5 = \left(1, 1+\sqrt{2}, 1+\sqrt{2}, 1\right)$	a_4
$L(6)$	$[-11-8\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	$a_1 = (0,-1,1,0) \quad a_2 = (0,0,-1,1)$ $a_3 = (0,0,0,-1) \quad a_4 = \left(1, 2+\sqrt{2}, 2+\sqrt{2}, 1\right)$ $a_5 = \left(1, 2+2\cdot\sqrt{2}, 1, 0\right)$ $a_6 = \left(1, 2+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\right)$ $a_7 = \left(2+\sqrt{2}, 7+5\sqrt{2}, 3+3\sqrt{2}, 2+\sqrt{2}\right)$ $a_8 = \left(1+2\sqrt{2}, 8+5\sqrt{2}, 4+3\sqrt{2}, 3+2\sqrt{2}\right)$ $a_9 = \left(1+2\sqrt{2}, 8+6\sqrt{2}, 3+2\sqrt{2}, 2+2\sqrt{2}\right)$ $a_{10} = \left(2+3\sqrt{2}, 13+9\sqrt{2}, 7+5\sqrt{2}, 2+\sqrt{2}\right)$ $a_{11} = \left(4+2\sqrt{2}, 13+10\sqrt{2}, 9+6\sqrt{2}, 0\right)$ $a_{12} = \left(4+4\sqrt{2}, 19+14\sqrt{2}, 9+6\sqrt{2}, 8+6\sqrt{2}\right)$ $a_{13} = \left(4+4\sqrt{2}, 20+14\sqrt{2}, 11+8\sqrt{2}, 1\right)$ $a_{14} = \left(4+2\sqrt{2}, 14+10\sqrt{2}, 6+4\sqrt{2}, 5+4\sqrt{2}\right)$ $a_{15} = \left(4+3\sqrt{2}, 17+12\sqrt{2}, 8+5\sqrt{2}, 6+4\sqrt{2}\right)$ $a_{16} = \left(4+3\sqrt{2}, 17+12\sqrt{2}, 9+7\sqrt{2}, 1+\sqrt{2}\right)$ $a_{17} = \left(5+4\sqrt{2}, 22+15\sqrt{2}, 13+9\sqrt{2}, 1+\sqrt{2}\right)$	$\emptyset$
$L(7)$	$[-\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	$a_1 = (0,-1,1,0) \quad a_2 = (0,0,-1,1)$ $a_3 = (0,0,0,-1) \quad a_4 = \left(1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}, 0\right)$ $a_5 = \left(1+\sqrt{2}, 2+\sqrt{2}, 0, 0\right)$ $a_6 = \left(2+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\right)$	$\emptyset$
$L(8)$	$\begin{bmatrix} 2 & -1 & -\sqrt{2} \\ -1 & 2 & \sqrt{2}-1 \\ -\sqrt{2} & \sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix} \oplus [1]$	$a_1 = \left(0,\ 0,\ 0,\ -\sqrt{2}\right) \qquad a_2 = \left(0,\ 0,\ -\sqrt{2},\ 0\right)$ $a_3 = \left(0, -\frac{\sqrt{2}}{2},\ 1+\frac{\sqrt{2}}{2},\ 0\right)$ $a_4 = \left(1+\sqrt{2},\ 2+\sqrt{2},\ 0, 0\right)$ $a_5 = \left(1+\sqrt{2}, 0, 0, \sqrt{2}+2, \right)$ $a_6 = \left(2+\sqrt{2}, 2+\sqrt{2}, 0, 2+\sqrt{2}\right)$	a_6
$L(12)$	$[-7-5\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	$a_1 = (0,-1,1,0) \quad a_2 = (0,0,-1,1)$ $a_3 = (0,0,0,-1) \quad a_4 = \left(2-\sqrt{2}, 1+\sqrt{2}, 1, 0\right)$ $a_5 = \left(1, 1+\sqrt{2}, 1+\sqrt{2}, 1+\sqrt{2}\right)$	$\emptyset$

TABLE 4. Stably reflective Lorentzian lattices of signature $(3,1)$ over $\mathbb{Z}[\sqrt{2}]$. Here $B(L)$ denotes the set of "bad" reflections.

Proof. Notice that $L(3)$ is isomorphic to the lattice with coordinates

$$y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$$

and with inner product, given by the quadratic form

$$f(y) = -(2\sqrt{2} - 1)y_0^2 + y_1^2 + y_2^2 + y_3^2,$$

where

$$\sqrt{2}y_0 \in \mathbb{Z}[\sqrt{2}], \quad y_0 + \frac{y_0 + y_1}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}], \quad y_2, y_3 \in \mathbb{Z}[\sqrt{2}].$$

Our program `VinAl` finds 8 roots:

$$\begin{aligned}
&a_1 = (0,\, 0,\, 0,\, -1)\,, && a_2 = (0,\, 0,\, -1,\, 1)\,,\\
&a_3 = \left(0,\, -\sqrt{2},\, 0,\, 0\right), && a_4 = \left(1 + \sqrt{2},\, 3 + \sqrt{2},\, 0, 0\right),\\
&a_5 = \left(1 + \sqrt{2}/2,\, \sqrt{2}/2,\, \sqrt{2} + 1,\, 0\right), && a_6 = \left(\sqrt{2} + 1,\, \sqrt{2} + 1,\, \sqrt{2} + 1,\, 1\right),\\
&a_7 = \left(\sqrt{2} + 1,\, 1,\, \sqrt{2} + 1,\, \sqrt{2} + 1\right), && a_8 = \left(3\sqrt{2} + 4,\, 0,\, 4\sqrt{2} + 5,\, \sqrt{2} + 3\right).
\end{aligned}$$

It is sufficient to consider the group generated by "bad" reflections with respect to mirrors H_{a_4} and H_{a_8}. Since these mirrors are divergent, this subgroup is infinite. ■

Proposition 7.3. *The lattice* $L(4) = \begin{bmatrix} 2 & -1 - 2\sqrt{2} \\ -1 - 2\sqrt{2} & 2 \end{bmatrix} \oplus [1] \oplus [1]$ *is reflective, but not stably reflective.*

Proof. Notice that $L(4)$ is isomorphic to the lattice with coordinates

$$y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$$

and with inner product, given by the quadratic form

$$f(y) = -(5 + 4\sqrt{2})y_0^2 + y_1^2 + y_2^2 + y_3^2,$$

where

$$\sqrt{2}y_0 \in \mathbb{Z}[\sqrt{2}], \quad \frac{y_0 + y_1}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}], \quad y_2, y_3 \in \mathbb{Z}[\sqrt{2}].$$

Our program `VinAl` finds 8 roots

$$\begin{aligned}
&a_1 = (0,\, 0,\, 0,\, -1)\,, && a_2 = (0,\, 0,\, -1,\, 1)\,,\\
&a_3 = \left(0,\, -\sqrt{2},\, 0,\, 0\right), && a_4 = \left(1,\, 3 + \sqrt{2},\, 0, 0\right),\\
&a_5 = \left(\sqrt{2}/2,\, \sqrt{2}/2,\, \sqrt{2} + 1,\, 0\right), && a_6 = \left(1,\, 1,\, \sqrt{2} + 1,\, \sqrt{2} + 1\right),\\
&a_7 = \left(1,\, \sqrt{2} + 1,\, \sqrt{2} + 1,\, 1\right), && a_8 = \left(\sqrt{2} + 2,\, 0,\, 4\sqrt{2} + 5,\, \sqrt{2} + 3\right).
\end{aligned}$$

It is sufficient to consider the group generated by "bad" reflections with respect to mirrors H_{a_4} and H_{a_8}. Since these mirrors are divergent, this subgroup is infinite. ■

Proposition 7.4. *The lattice* $L(9) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -\sqrt{2} \end{bmatrix} \oplus [1]$ *is not stably reflective.*

Proof. Note that $L(9)$ is isomorphic to the lattice with coordinates $y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$ and with inner product, given by the quadratic form $(y) = -\sqrt{2}\, y_0^2 + (3+\sqrt{2})y_1^2 + y_2^2 + y_3^2$, where $\sqrt{2}\, y_1 \in \mathbb{Z}[\sqrt{2}]$, $\frac{y_1+y_2}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}]$, $y_1 - y_0,\ y_3 \in \mathbb{Z}[\sqrt{2}]$.

Our program `VinAl` finds 9 first roots

$$\begin{aligned}
a_1 &= (0,\,0,\,0,\,-1)\,, & a_2 &= \left(0,\,0,\,-\sqrt{2},\,0\right),\\
a_3 &= \left(0,\,-\sqrt{2},\,0,\,0\right), & a_4 &= \left(1+\sqrt{2},0,0,2+\sqrt{2}\right),\\
a_5 &= \left(1+\sqrt{2},0,2+\sqrt{2},0\right), & a_6 &= \left(2+\sqrt{2},0,2+\sqrt{2},2+\sqrt{2}\right),\\
a_7 &= \left(1+\sqrt{2},1,1+\sqrt{2},0\right), & a_8 &= \left(2+3\sqrt{2}/2,1+\sqrt{2}/2,1+\sqrt{2}/2,2+\sqrt{2}\right),
\end{aligned}$$
$$a_9 = \left(5+4\sqrt{2},2+\sqrt{2},0,5+4\sqrt{2}\right).$$

It is sufficient to consider the subgroup generated by "bad" reflections with respect to mirrors H_{a_3}, H_{a_6} and H_{a_9}. Since the mirrors H_{a_6} and H_{a_9} are divergent, this subgroup is infinite. ∎

Proposition 7.5. *The lattice* $L(10) = \begin{bmatrix} 2 & -1-\sqrt{2} \\ -1-\sqrt{2} & 2 \end{bmatrix} \oplus [2+\sqrt{2}] \oplus [1]$ *is not stably reflective.*

Proof. Note that $L(10)$ is isomorphic to the lattice with coordinates $y = (y_0, y_1, y_2, y_3) \in \mathbb{Q}^4[\sqrt{2}]$ and with inner product, given by the quadratic form

$$f(y) = -(1+2\sqrt{2})\, y_0^2 + (2+\sqrt{2})y_1^2 + y_2^2 + y_3^2,$$

where $\sqrt{2}\, y_0 \in \mathbb{Z}[\sqrt{2}]$, $\frac{y_0+y_2}{\sqrt{2}} \in \mathbb{Z}[\sqrt{2}]$, $y_1, y_3 \in \mathbb{Z}[\sqrt{2}]$.

Our program `VinAl` finds 8 first roots

$$\begin{aligned}
a_1 &= (0,\,0,\,0,\,-1)\,, & a_2 &= \left(0,\,0,\,-\sqrt{2},\,0\right),\\
a_3 &= \left(0,\,-\sqrt{2},\,0,\,0\right), & a_4 &= \left(\sqrt{2}/2,\,0,\,2+\sqrt{2}/2,0\right),\\
a_5 &= \left(1+\sqrt{2}/2,\,0,1+\sqrt{2}/2,\,\sqrt{2}+2\right), & a_6 &= \left(1+\sqrt{2},\,1+\sqrt{2},\,\sqrt{2}+1,0\right),\\
a_7 &= \left(2+\sqrt{2},2+\sqrt{2},0,2+\sqrt{2}\right), & a_8 &= \left(2+\sqrt{2},\,1+2\sqrt{2},0,0\right).
\end{aligned}$$

It is sufficient to consider the group generated by "bad" reflections with respect to mirrors H_{a_3} and H_{a_8}. Since these mirrors are divergent, this subgroup is infinite. ∎

Proposition 7.6. *The lattices* $L(13)$, $L(14)$ *and* $L(15)$ *are not reflective.*

Proof. Non-reflectivity of these lattices is determined by the method of infinite symmetry, described in §5.3.2. The implementation of this method is avalaible here https://github.com/nvbogachev/VinAlg-Z-sqrt-2-/blob/master/Infinite-Symm.py ∎

Thus, the lattices $L(1)$, $L(2)$, $L(5)$, $L(6)$, $L(7)$, $L(8)$, and $L(12)$, are stably reflective. This completes the proof of Theorem C. □

The next step in this direction can be finding all stably reflective Lorentzian $\mathbb{Z}[\sqrt{2}]$-lattices of signature $(3,1)$ or over other rings of integers.

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